JSSAC Bulletin of JSSAC (2019)
Vol. 25, No. 2, pp. 2 - 11

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Completing the Computation of the Explicit Formula for the Circumradius of Cyclic Octagons*

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(Received April 3, 2019 Accepted May 15, 2019)

Abstract

This paper presents the results of computing the circumradius formula for cyclic octagons given by the lengths of the sides. Continuing with the author's recent paper in 2018, we have finally succeeded in computing all the coefficients of the octagon formula, which has 845,027 terms in the form of elementary symmetric polynomials. We have also fixed the explicit formula with 7,639,619,878 terms in the expression by the lengths of sides. In this study, we newly adopted the method of numerical interpolation, adding to the method of elimination by resultants, which has been applied in the author's preceding papers. We have found that modular algorithms based on the Chinese remainder theorem are indispensable in our problems for solving systems of linear equations over the integers.

1 Introduction

In this study, we consider a classic problem in Euclidean geometry for cyclic polygons; that is, polygons inscribed in a circle. In particular, we focus on computing the circumradius R of cyclic n-gons given by the lengths of sides $a_1, a_2, \ldots, a_n$. In a previous paper [5], the author discussed the computation and analysis of explicit formulae for the circumradii of cyclic heptagons and octagons. However, in the octagon formula, 14 out of 39 coefficients remained uncompleted because of the CPU time required for their computation. On the other hand, analysis of the formulae by an investigation in terms of total degrees gave an insight into the form and structure of the octagon formula. Hence, this study was conducted with the aim of obtaining the completely expanded form of the octagon formula, applying both of the following two methods:

(1) continuation of the computation using resultants,
(2) another approach using numerical interpolation.

*This work was supported by a Grant-in-Aid for Scientific Research (25330006) from the Japan Society for the Promotion of Science (JSPS).

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Since Robbins [9] showed the "area formula (Heron polynomial)" for cyclic pentagons and conjectured their degrees, the authors of several reports including [1] [2] [7] [8] [10] [11] have studied this problem of the area. Some of them have also discussed the "circumradius formula," such as Robbins [9], Varfolomeev [11], Svrtan et al. [10], and Pech [8], but it seems that an explicit formula for $n > 5$ has not been shown in these papers. As a related work, the author derived an "integrated formula" for the relation of circumradius R and area S for $n = 5, 6$ in [4].

In contrast, this paper focuses on the "circumradius formula for cyclic octagons," which has not yet been explicitly computed. The computation itself is very simply realized by resultants. If we already have the circumradius formulae $\Phi_6^{(+)}(a_i;\ y)$ and $\Phi_4^{(+)}(a_i;\ y)$ for a (convex) cyclic hexagon and quadrilateral, we can directly compute the octagon formula using the following resultant:

$$\Phi_8^{(+)}(a_i;\ y) := \mathrm{Res}_d(\Phi_6^{(+)}(a_1, a_2, a_3, a_4, a_5, d;\ y),\ \Phi_4^{(+)}(d, a_6, a_7, a_8;\ y))/y^6, \tag{1}$$

where $y = R^2$. If we could compute the elimination by resultants efficiently, this equation would be easily solved. However, the size of polynomials for $n \geq 7$ becomes so huge that we need much more consideration than for a straight computation.

To the best of our knowledge, there exist no reports in which the circumradii for $n \geq 6$ are explicitly computed, other than the author's previous papers [3][5]. We note that some partial results of this study have already been submitted by the author [6]. Now, however, we have finally completed the computation of the octagon formula for the first time, both the expression by the lengths a_i of sides and their elementary symmetric polynomials s_i:

$$\Phi_8^{(+)}(a_i;\ y) = P_{38}y^{38} + \cdots + P_1 y + P_0 \qquad \text{(7,639,619,878 terms, approx. 160 GB)} \quad (y = R^2,\quad P_i \in \mathbf{Z}[a_1, \ldots, a_8]), \tag{2}$$

$$F_8^{(+)}(s_i;\ y) = \tilde{P}_{38}y^{38} + \cdots + \tilde{P}_1 y + \tilde{P}_0 \qquad \text{(845,027 terms, approx. 19 MB)} \quad (y = R^2,\quad \tilde{P}_i \in \mathbf{Z}[s_1, \ldots, s_7, \sqrt{s_8}]), \tag{3}$$

where $s_1 = a_1^2 + a_2^2 + \cdots + a_8^2$, $s_2 = a_1^2 a_2^2 + \cdots, \cdots$, $s_7 = a_1^2 \cdots a_7^2 + \cdots$, $\sqrt{s_8} = a_1 \cdots a_8$. It seems that these polynomials have not been shown elsewhere so far.

2 Robbins' theorem and previously known results up to $n = 7$

For a given cyclic n-gon with the lengths of sides $a_1, \ldots, a_n$, we define its circumradius formula as a polynomial $\Phi_n(a_1, \ldots, a_n;\ R^2)$ where all the possible circumradii R are contained as its roots.

The degrees of defining polynomials $\Phi_n(a_i;\ y)$, where $y = R^2$, is given by the following Robbins' conjecture [9], which was proved by Fedorchuk and Pak [1]. Let

$$k_m := \frac{2m+1}{2}\binom{2m}{m} - 2^{2m-1} = \sum_{j=0}^{m-1}(m-j)\binom{2m+1}{j}; \tag{4}$$

that is, let $k_i := 1, 7, 38, 187, 874, \ldots$ $(i = 1, 2, 3, 4, \ldots)$. Then,

- the degree in y of $\Phi_{2m+1}(a_i;\ y)$ is k_m, and
- the degree in y of $\Phi_{2m+2}^{(\pm)}(a_i;\ y)$ is $2k_m$, where $\Phi_{2m+2}^{(\pm)}$ is factored into the product of two polynomials, $\Phi_{2m+2}^{(+)}$ and $\Phi_{2m+2}^{(-)}$, with each degree k_m.

In the following subsections, we summarize the previously known results for $n = 3, \ldots, 7$ according to the author's previous paper [5].

2.1 Circumradius of a triangle ($n = 3$)

Every triangle with side lengths a_1, a_2, and a_3 has a circumcircle, and its radius R is given by the classical formula of Heron. Converting it into a polynomial expression, and letting $y := R^2$, we obtain the defining polynomial of y for a triangle as

$$\Phi_3(a_1, a_2, a_3;\ y) := \left(a_1^4 + a_2^4 + a_3^4 - 2(a_1^2a_2^2 + a_2^2a_3^2 + a_3^2a_1^2)\right) y + a_1^2a_2^2a_3^2. \tag{5}$$

Using elementary symmetric polynomials in a_i^2, we rewrite the above result as

$$F_3(s_1, s_2, s_3;\ y) := (s_1^2 - 4s_2)y + s_3, \tag{6}$$

where $s_1 = a_1^2 + a_2^2 + a_3^2$, $s_2 = a_1^2a_2^2 + a_2^2a_3^2 + a_3^2a_1^2$, and $s_3 = a_1^2a_2^2a_3^2$.

2.2 Circumradius of a cyclic quadrilateral ($n = 4$)

We have the classic result of Brahmagupta for a "convex" cyclic quadrilateral, and from its polynomial expression, we define the circumradius formula as

$$\begin{aligned}\Phi_4^{(+)}(a_i;\ y) := \Big((a_1^4 + a_2^4 + a_3^4 + a_4^4) - 2(a_1^2a_2^2 + a_1^2a_3^2 + a_1^2a_4^2 + a_2^2a_3^2 + a_2^2a_4^2 + a_3^2a_4^2) - 8a_1a_2a_3a_4\Big) y \\ + (a_1^2a_2^2a_3^2 + a_1^2a_2^2a_4^2 + a_1^2a_3^2a_4^2 + a_2^2a_3^2a_4^2) + (a_1^2 + a_2^2 + a_3^2 + a_4^2)a_1a_2a_3a_4.\end{aligned} \tag{7}$$

Using elementary symmetric polynomials in a_i^2, we rewrite the above result as

$$F_4^{(+)}(s_i;\ y) := (s_1^2 - 4s_2 - 8\sqrt{s_4})y + (s_3 + s_1\sqrt{s_4}), \tag{8}$$

where $s_1 = a_1^2 + a_2^2 + a_3^2 + a_4^2$, $s_2 = a_1^2a_2^2 + \cdots$, $s_3 = a_1^2a_2^2a_3^2 + \cdots$, and $\sqrt{s_4} = a_1a_2a_3a_4$, which is used as an auxiliary to $s_4 = a_1^2a_2^2a_3^2a_4^2$.

We should note that, letting $a_4 := -a_4$ or $\sqrt{s_4} := -\sqrt{s_4}$, we obtain the other polynomials for "non-convex" quadrilaterals:

$$\Phi_4^{(-)}(a_1, a_2, a_3, a_4;\ y) := \Phi_4^{(+)}(a_1, a_2, a_3, -a_4;\ y), \tag{9}$$

$$F_4^{(-)}(s_i;\ y) := (s_1^2 - 4s_2 + 8\sqrt{s_4})y + (s_3 - s_1\sqrt{s_4}). \tag{10}$$

Introducing the notion of "crossing parity" ε [9][2], where ε is 0 for a triangle, +1 for a convex quadrilateral, and −1 for a non-convex quadrilateral, the circumradius formulae in $y = R^2$ for $n = 3, 4$ in Eqs. (6), (8), and (10) are written in the following unified form:

$$F_{3,4}(s_i;\ y) := (s_1^2 - 4s_2 - \varepsilon \cdot 8\sqrt{s_4})y + (s_3 + \varepsilon \cdot s_1\sqrt{s_4}). \tag{11}$$

2.3 Circumradius of a cyclic pentagon ($n = 5$)

We divide a cyclic pentagon with side lengths $\{a_1, \ldots, a_5\}$ by a diagonal of length d, into a cyclic quadrilateral of sides $\{a_1, a_2, a_3, d\}$ and a triangle of sides $\{d, a_4, a_5\}$.

Since this quadrilateral and triangle have circumradius R in common, the cyclic pentagon formula should be obtained if the diagonal d is eliminated from the formulae of Brahmagupta and Heron. Specifically, we need to compute the following resultant:

$$\begin{aligned}\Phi_5(a_i;\ y) &:= \mathrm{Res}_d(\Phi_4^{(+)}(a_1, a_2, a_3, d;\ y),\ \Phi_3(d, a_4, a_5;\ y))/y \\ &= A_7y^7 + A_6y^6 + A_5y^5 + A_4y^4 + A_3y^3 + A_2y^2 + A_1y + A_0 \qquad \text{(2,922 terms)} \\ &\qquad \left(y = R^2, \quad A_i \in \mathbf{Z}[a_1^2, \ldots, a_5^2]\right).\end{aligned} \tag{12}$$

It should also be helpful to reduce the expression for the pentagon case, using the elementary symmetric polynomials $s_1 = a_1^2 + \cdots + a_5^2, \ldots, s_5 = a_1^2 \cdots a_5^2$:

$$F_5(s_i;\ y) = \tilde{A}_7 y^7 + \tilde{A}_6 y^6 + \cdots + \tilde{A}_1 y + \tilde{A}_0 \qquad \text{(81 terms)} \quad \left(\tilde{A}_i \in \mathbf{Z}[s_1, \ldots, s_5]\right). \tag{13}$$

2.4 Circumradius of a cyclic hexagon ($n = 6$)

Dividing a (convex) cyclic hexagon into two (convex) quadrilaterals, we compute the defining polynomial for the circumradius by resultants:

$$\begin{aligned} \Phi_6^{(+)}(a_i;\ y) &:= \mathrm{Res}_d(\Phi_4^{(+)}(a_1, a_2, a_3, d;\ y),\ \Phi_4^{(+)}(d, a_4, a_5, a_6;\ y))/y \\ &= B_7 y^7 + B_6 y^6 + \cdots + B_1 y + B_0 \qquad \text{(19,449 terms, approx. 580 KB)} \\ &\qquad \left(y = R^2, \quad B_i \in \mathbf{Z}[a_1, \ldots, a_6]\right). \end{aligned} \tag{14}$$

Using the elementary symmetric polynomials $s_1 = a_1^2 + \cdots + a_6^2, \ldots, s_5 = a_1^2 a_2^2 a_3^2 a_4^2 a_5^2 + \cdots$, $\sqrt{s_6} = a_1 \cdots a_6$, we rewrite Eq. (14) into a simpler form:

$$F_6^{(+)}(s_i;\ y) := \tilde{B}_7 y^7 + \tilde{B}_6 y^6 + \cdots + \tilde{B}_1 y + \tilde{B}_0 \qquad \text{(224 terms)} \quad \left(\tilde{B}_i \in \mathbf{Z}[s_1, \ldots, s_5,\ \sqrt{s_6}]\right). \tag{15}$$

The counterparts for hexagons of the other group without a convex one are obtained by simple substitution from Robbins' theorem:

$$\Phi_6^{(-)}(a_1, \ldots, a_5, a_6;\ y) := \Phi_6^{(+)}(a_1, \ldots, a_5, -a_6;\ y), \tag{16}$$

$$F_6^{(-)}(s_1, \ldots, s_5,\ \sqrt{s_6};\ y) := F_6^{(+)}(s_1, \ldots, s_5, -\sqrt{s_6};\ y). \tag{17}$$

Since we also have the relation $F_5(s_1, \ldots, s_5;\ y) = F_6^{(+)}(s_1, \ldots, s_5, 0;\ y)$, we can express F_5, $F_6^{(+)}$, and $F_6^{(-)}$ uniformly as polynomial $F_{5,6}(s_1, \ldots, s_5, \varepsilon\sqrt{s_6};\ y)$ similarly to Eq. (11), using the crossing parity ε.

2.5 Circumradius of a cyclic heptagon ($n = 7$)

In our previous paper [5], after comparative experiments, we concluded that the following method of resultant computation seems to be quite practical from the viewpoint of CPU time and memory consumption. In this formulation, we divide a cyclic heptagon into a pentagon and a convex quadrilateral by another diagonal d, and compute the resultant into the expanded form:

$$\begin{aligned} \Phi_7(a_i;\ y) &:= \mathrm{Res}_d(\Phi_5(a_1, a_2, a_3, a_4, d;\ y),\ \Phi_4^{(+)}(d, a_5, a_6, a_7;\ y))/y^6 \\ &= C_{38} y^{38} + \cdots + C_1 y + C_0 \qquad \text{(337,550,051 terms, approx. 7,407 MB)} \\ &\qquad \left(y = R^2, \quad C_i \in \mathbf{Z}[a_1^2, \ldots, a_7^2]\right). \end{aligned} \tag{18}$$

We also succeeded in converting $\Phi_7(a_i;\ y)$ into the form of elementary symmetric polynomials:

$$F_7(s_i;\ y) = \tilde{C}_{38} y^{38} + \cdots + \tilde{C}_1 y + \tilde{C}_0 \qquad \text{(199,695 terms)} \quad \left(\tilde{C}_i \in \mathbf{Z}[s_1, \ldots, s_7]\right). \tag{19}$$

For reference, the area formula ($n = 7, 8$) reported by Maley et al. [2] has the following form:

$$\begin{aligned} G_7(s_i;\ x) &= x^{38} + \tilde{M}'_{37} x^{37} + \cdots + \tilde{M}'_1 x + \tilde{M}'_0 \qquad \text{(955,641 terms)} \\ &\qquad \left(x = (4S)^2, \quad \tilde{M}'_i \in \mathbf{Z}[s_1, \ldots, s_7]\right), \\ G_8^{(+)}(s_i;\ x) &= x^{38} + \tilde{M}_{37} x^{37} + \cdots + \tilde{M}_1 x + \tilde{M}_0 \qquad \text{(3,248,266 terms)} \\ &\qquad \left(x = (4S)^2, \quad \tilde{M}_i \in \mathbf{Z}[s_1, \ldots, s_7,\ \sqrt{s_8}]\right), \end{aligned} \tag{20}$$

where S is the area of the polygon and we have $\tilde{M}'_j = \tilde{M}_j \mid_{\sqrt{s_8}=0}$ $(0 \le j \le 37)$.

3 Circumradius of a cyclic octagon ($n = 8$): Method 1

In this study, the Maple 2016 and 2017 computer algebra systems were used in the following two environments. Since the computations were not carried out in a unified environment, the data for CPU times described below are only for reference and are not necessarily comparable.

Machine A Windows, Xeon (8 core, 2.93 GHz) ×2, 192 GB RAM,
Machine B Linux, Xeon (8 core, 2.6 GHz) ×2, 256 GB RAM.

3.1 Method of expansion of resultant

We review the algorithm described in the author's previous paper [5]. Dividing an octagon into a (convex) hexagon and a (convex) quadrilateral, we compute the resultant in Eq. (1) stepwise.

First, we collect the coefficients of the two polynomials in d:

$$\begin{cases} \Phi_6^{(+)}(a_1,a_2,a_3,a_4,a_5,d;\ y) = y^7 d^{16} - a_1a_2a_3a_4a_5 y^5 d^{15} + u_{14}d^{14} + \cdots + u_1 d + u_0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (u_j \in \mathbf{Z}[a_1,\ldots,a_5,y]), \\ \Phi_4^{(+)}(d,a_6,a_7,a_8;\ y) = yd^4 + a_6a_7a_8 d^3 + (\cdots)d^2 + (\cdots)d + (\cdots) \qquad (19\text{ terms}), \end{cases} \tag{21}$$

where $\Phi_6^{(+)}$ originally has 19,449 terms.

Second, we compute the resultant of these polynomials, regarding $u_0,\ldots,u_{14}$ as independent new variables. Then, we obtain the intermediate form of the resultant polynomial:

$$R(u_0,u_1,\ldots,u_{14},a_1,\ldots,a_8;\ y) := \mathrm{Res}_d(\Phi_6^{(+)},\Phi_4^{(+)}). \tag{22}$$

Third, we substitute the original coefficient $u_j(a_1,\ldots,a_5,y)$ in $\Phi_6^{(+)}$ into each u_j, and obtain the following polynomial:

$$\bar{R}(a_1,\ldots,a_8;\ y) = \bar{P}_{38}y^{44} + \cdots + \bar{P}_0 y^6. \tag{23}$$

At this point, the $\bar{P}_i$'s have not yet been expanded or simplified and it is difficult to observe their explicit expressions. Finally, if we succeed in expanding each coefficient $\bar{P}_i$, we obtain the circumradius formula $\Phi_8^{(+)}(a_i;\ y)$ in Eq. (1). The current status of computation, updated from the previous paper [5], is expressed as follows:

$$\Phi_8^{(+)}(a_i;\ y) = P_{38}y^{38} + \cdots + P_{28}y^{28} + \left(\bar{P}_{27}y^{27} + \cdots + \bar{P}_{15}y^{15}\right) + P_{14}y^{14} + \cdots + P_0, \tag{24}$$

where coefficients $P_{27},\ldots,P_{15}$ with much larger sizes have not yet been obtained in expanded form. A summary of the number of terms is shown in Table 1, together with the results obtained by another method described later.

The expansion of each coefficient $\bar{P}_i$ needs a large memory allocation and often fails. In order to avoid memory overflow, we need to divide the procedure into a number of smaller problems, which requires much more CPU time.

For example, the size of coefficient P_{14} is approximately 8,834 MB in Maple file format (*.m). As a result, the expansion of $\bar{P}_{14}$ required 1,004 days of CPU time in total (with 178 jobs, on Machine B described above), which is the latest result since the previous paper [5]. Hence, it is unlikely that the remaining computations will be completed in the near future.

When we obtained the coefficient P_i in expanded form, we converted it into an expression in the form of elementary symmetric polynomials, applying the algorithm proposed in [5]. Currently, we have obtained the explicit form of coefficients except $\tilde{P}_{27},\ldots,\tilde{P}_{15}$, as follows:

$$F_8^{(+)}(s_i;\ y) = \tilde{P}_{38}y^{38} + \cdots + \tilde{P}_{28}y^{28} + \left(\bar{P}_{27}y^{27} + \cdots + \bar{P}_{15}y^{15}\right) + \tilde{P}_{14}y^{14} + \cdots + \tilde{P}_0. \tag{25}$$

3.2 Analysis of the forms of circumradius formulae $\Phi_8^{(+)}(a_i;\ y)$ and $F_8^{(+)}(a_i;\ y)$

In the study described in the author's previous paper [5], we investigated the shapes of circumradius formulae by focusing on the degrees in each coefficient, and found the regularity of their distribution.

First, we introduce the notion of the total degree of a power product in a_i^2's:

$$\text{t-deg}\left(a_1^{2m_1}a_2^{2m_2}\cdots a_n^{2m_n}\right) := m_1 + m_2 + \cdots + m_n. \tag{26}$$

Under this definition, the total degree of the form of elementary symmetric polynomials with n variables is given by

$$\text{t-deg}\left(s_1^{m_1}s_2^{m_2}\cdots s_n^{m_n}\right) = m_1 + 2m_2 + \cdots + nm_n, \tag{27}$$

noting that we define, if n is even, t-deg$(\sqrt{s_n}) = n/2$, where $\sqrt{s_n} = a_1a_2\cdots a_n$.

Analyzing the octagon formulae $\Phi_8^{(+)}(a_i;\ y)$ and $F_8^{(+)}(a_i;\ y)$ in Eqs. (24) and (25), we observed the number of terms and the total degrees of the coefficients as shown in Table 1.

The regularity of the distribution of degrees makes it possible to readily estimate the forms of $\tilde{P}_i$ $(i = 15, \ldots, 27)$, the expanded forms of which we have not yet obtained. For example, $\tilde{P}_{20}$ should have t-deg 50 and degree 12 in $\sqrt{s_8}$. Therefore, it should have the following form:

$$\tilde{P}_{20} = u_0(s_1, \ldots, s_7) + u_1(s_1, \ldots, s_7)\sqrt{s_8} + \cdots + u_{12}(s_1, \ldots, s_7)\sqrt{s_8}^{12}, \tag{28}$$

where u_j is homogeneous with t-deg$(u_j) = 50 - 4j$ $(j = 0, \ldots, 12)$. In particular, $u_0(s_1, \ldots, s_7)$ should be identical with coefficient $\tilde{C}_{20}$ of the heptagon formula $F_7(s_i;\ y)$ in Eq. (19).

Based on these observations, we have developed a new approach using numerical interpolation.

4 Circumradius of a cyclic octagon ($n = 8$): Method 2

4.1 Method of numerical interpolation

We consider directly computing each coefficient polynomial $\tilde{P}_d(s_i)$ of y^d $(0 \le d \le 38)$ in Eq. (25), in the form of elementary symmetric polynomials.

First, we assume that we have already obtained the circumradius formulae for cyclic quadrilaterals and hexagons. When randomly chosen integers $\alpha_1, \ldots, \alpha_8$ are substituted into Eqs. (7) and (14), we can easily compute the resultant

$$\begin{aligned} \Phi_8^{(+)}(\alpha_i;\ y) &:= \mathrm{Res}_d(\Phi_6^{(+)}(\alpha_1, \ldots, \alpha_5, d;\ y),\ \Phi_4^{(+)}(d, \alpha_6, \alpha_7, \alpha_8;\ y))/y^6 \\ &= w_{38}(\alpha_i)y^{38} + \cdots + w_d(\alpha_i)y^d + \cdots + w_0(\alpha_i). \end{aligned} \tag{29}$$

Next, we compute each coefficient polynomial $\tilde{P}_d(s_i)$ in Eq. (25) using the following procedure.

(1) We search for all the 8-tuples of integers $(e_1, \ldots, e_7, e_8)$ in the range of $0 \le e_j \le (70 - d)/j$, so that $e_1 + 2e_2 \cdots + 7e_7 + 4e_8 = 70 - d$ is satisfied. We let the number of found 8-tuples be N.

(2) We generate N monomials $m_k = s_1^{e_1^{(k)}} \cdots s_7^{e_7^{(k)}} \sqrt{s_8}^{e_8^{(k)}}$ $(k = 1, \ldots, N)$. We note that m_k is also a polynomial in a_i through the relations $s_1 = a_1^2 + \cdots + a_8^2, \ldots, \sqrt{s_8} = a_1 \cdots a_8$.

(3) We let $f(a_i) = c_1m_1 + \cdots + c_Nm_N$ using indeterminate coefficients $c_1, \ldots, c_N$.

(4) We choose a set of random integers $(\alpha_1, \ldots, \alpha_8)$ and substitute them into $f(a_i)$. On the other hand, we compute $w_d(\alpha_i)$ according to Eq. (29). Then, we have a linear equation over the integers $f(\alpha_i) = w_d(\alpha_i)$, with indeterminate $c_1, \ldots, c_N$.

(5) We choose another set of random integers $(\alpha'_1, \dots, \alpha'_8)$, and similarly compute a linear equation $f(\alpha'_i) = w_d(\alpha'_i)$. If we choose "linearly independent" N sets of 8-tuples, we have a system of linear equations over the integers $A\boldsymbol{x} = \boldsymbol{b}$ with N indeterminate.
(6) Solving $A\boldsymbol{x} = \boldsymbol{b}$ and using its solution $\boldsymbol{x} = [\gamma_1, \dots, \gamma_N]^T$, we obtain the coefficient polynomial $\tilde{P}_d(s_i) = \gamma_1 m_1 + \cdots + \gamma_N m_N$.

In our study, using the "nextprime" Maple function, we generated the sequence of prime numbers $p_1 = 101, p_2 = 103, \dots$ and let the "linearly independent" set of lengths of sides be

$$(p_1, p_2, \dots, p_8),\ (p_2, p_3, \dots, p_9),\ \dots,\ (p_N, p_{N+1}, \dots, p_{N+7}). \tag{30}$$

We confirmed that all the coefficient matrix A were not singular throughout the computation.

We attempted computations by implementing the above-mentioned procedure. It was easy to solve the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ when the matrix size N was small. However, it seems impossible to compute all the coefficients $\tilde{P}_{27}, \dots, \tilde{P}_{15}$, which have not been obtained in Eq. (25), because the size of the matrix grows sequentially in this order. Therefore, we have considered the following three devices.

4.2 Device 1: Using the already obtained results $F_7(s_i;\ y)$

We note that $F_7(s_i;\ y)$ in Eq. (19) is obtained by substituting $\sqrt{s_8} := 0$ to $F_8^{(+)}(s_i;\ y)$ in Eq. (25). In the following, we illustrate the procedure using the example of case $d = 20$, in order to show the effect of the device.

The coefficient polynomial $\tilde{P}_{20}$ should have the form shown in Eq. (28), and $u_0(s_1, \dots, s_7)$ have already been computed as the coefficient of y^{20} in $F_7(s_i;\ y)$, which has $9,577$ terms. Therefore, we have only to compute the other monomials with positive degrees in $\sqrt{s_8}$.

(1) We search for all the 8-tuples of integers $(e_1, \dots, e_7, e_8)$ in the range of

$$0 \le e_1 \le 50/1,\ \ \dots,\ \ 0 \le e_7 \le 50/7,\ \ 1 \le e_8 \le 50/4, \tag{31}$$

so that $e_1 + 2e_2 \cdots + 7e_7 + 4e_8 = 50$ is satisfied. We let the number of found sets be $\tilde{N} := 32,255$, whereas the number of original candidates was $N = 50,393$.
(2) Generating $\tilde{N}$ monomials $m_k = s_1^{e_1^{(k)}} \cdots s_7^{e_7^{(k)}} \sqrt{s_8}^{e_8^{(k)}}$ $(k = 1, \dots, \tilde{N})$, we let $\tilde{f}(a_i) = c_1 m_1 + \cdots + c_{\tilde{N}} m_{\tilde{N}}$ using indeterminate coefficients $c_1, \dots, c_{\tilde{N}}$.
(3) We choose a set of random integers $(\alpha_1, \dots, \alpha_8)$ and substitute them into $\tilde{f}(a_i)$. On the other hand, we compute $w_{20}(\alpha_i)$ according to Eq. (29). Moreover, we substitute $\alpha_i, \dots, \alpha_7$ into $u_0(s_i)$, letting $\alpha_8 := 0$. Then, we have a linear equation over the integers $\tilde{f}(\alpha_i) = w_{20}(\alpha_i) - u_0(\alpha_i)$ with indeterminate $c_1, \dots, c_{\tilde{N}}$.
(4) If we choose "linearly independent" $\tilde{N}$ sets of 8-tuples, we have a system of linear equations over the integers $A\boldsymbol{x} = \boldsymbol{b}$ with $\tilde{N}$ indeterminate.
(5) Solving $A\boldsymbol{x} = \boldsymbol{b}$ and using its solution $\boldsymbol{x} = [\gamma_1, \dots, \gamma_{\tilde{N}}]^T$, we obtain the coefficient polynomial $\tilde{P}_{20}(s_i) = u_0 + \gamma_1 m_1 + \cdots + \gamma_{\tilde{N}} m_{\tilde{N}}$.

4.3 Device 2: Excluding unnecessary monomials by preprocessing

Although the matrix size of the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ is reduced to $\tilde{N} = 32,255$ by the device described above, we still found it difficult to solve them directly over the integers.

Therefore, we preprocessed the equation using modular arithmetic, to find the zero elements in its solution vector $\boldsymbol{x}$.

(1) We solve the equation $A\boldsymbol{x} = \boldsymbol{b}$ over $\mathbf{Z}_p$ and compute the solution:

$$\boldsymbol{x} \equiv [\ldots, \star, 0, \star, \ldots, \star, 0, 0, \ldots]^T \pmod{p}, \tag{32}$$

where $\star$'s are non-zero elements and we let the number of them be N'. Although this step is probabilistic, the result should be mostly true when the matrix $A \pmod{p}$ turns out to be regular. If it seems unsure, we should compute again using another prime number p'.

(2) The columns in matrix A corresponding to $\star$'s mean "necessary monomials." Using these columns, we extract a system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ with N' indeterminate.

In the case of $d = 20$, the size of the matrix was reduced to $N' = 24,713$. Adding the $9,577$ terms in the coefficient of y^{20} in $F_7(s_i;\ y)$, we can determine that the total number of monomials in the coefficient of y^{20} in $F_8^{(+)}(s_i;\ y)$ should be $34,290$.

4.4 Device 3: Solving systems of linear equations by modular algorithms

In the case of $d = 20$, the preprocess mentioned above gives a system of linear equations $A'\boldsymbol{x}' = \boldsymbol{b}'$ with matrix size $N' = 24,713$. However, it still seems impossible to compute the solution $\boldsymbol{x}'$ directly over the integers, because of the memory requirement.

Therefore, we applied modular algorithms, which are often used for various types of larger sized problems over the integers, to reduce memory consumption.

(1) We solve the equation $A'\boldsymbol{x}' = \boldsymbol{b}'$ with mod p_j $(j = 1, 2, \ldots, t)$, and obtain the solutions $\boldsymbol{x}'_{(1)}$, $\boldsymbol{x}'_{(2)}$, ..., $\boldsymbol{x}'_{(t)}$. Then, using the Chinese remainder theorem, we reconstruct the solution $\boldsymbol{x}'$ over the integers. We apply Newton's interpolation formula, adding modulo p_j incrementally.

(2) We prepared another relation $\boldsymbol{a}^T\boldsymbol{x} = b$ using the other set of integers, to confirm the recovery of the solution $\boldsymbol{x}$ over $\mathbf{Z}$. Actually, the coefficients $\tilde{P}_{38}, \ldots \tilde{P}_{25}$ needed three $(t = 3)$ moduli, and the coefficients $\tilde{P}_{24}, \ldots \tilde{P}_{11}$ needed only two $(t = 2)$ moduli for their recovery.

We took moduli among the prime numbers such that $p_j < 2^{32} - 1$. In the Maple computer algebra system, they are efficiently processed using hardware operation over 64-bit integers. As a result, we succeeded in computing the coefficient polynomials $\tilde{P}_{38}, \ldots \tilde{P}_{11}$ by the method of numerical interpolation. In comparison with the expansion of resultant, the computation of $\tilde{P}_{14}$ required 62 days of CPU time in total, on Machine A described in section 3.

5 Concluding remarks

Applying two types of algorithms, we have completed the computation of circumradius formulae for cyclic octagons $\Phi_8^{(+)}(a_i;\ y)$ and $F_8^{(+)}(s_i;\ y)$ in Eqs. (2) and (3), for the first time. A summary of the shapes of the formulae and applied methods is shown in Table 1.

1. Expanding the resultant, we have succeeded in the computation of 26 coefficient polynomials out of 39; that is, $\tilde{P}_{38}, \ldots, \tilde{P}_{28}$ and $\tilde{P}_{14}, \ldots, \tilde{P}_0$.
2. We have newly proposed an algorithm using numerical interpolation, and created three devices to solve the huge system of linear equations. As a result, we have succeeded in the computation of $\tilde{P}_{38}, \ldots, \tilde{P}_{11}$. These results contain the 13 coefficient polynomials $\tilde{P}_{27}, \ldots, \tilde{P}_{15}$, which are extremely difficult to compute by the expansion of resultants.
3. We have obtained the explicit form of the circumradius formula for cyclic octagons with 845,027 terms in the expression using elementary symmetric polynomials. Expanding them

conversely, we have found that the formula contains 7,639,619,878 terms in the expression by lengths of sides.

We have confirmed the correctness of the result in the following two ways.

Check 1 If we consider the equilateral case, we have the following structure:

$$F_8^{(+)}(y) = \begin{cases} 2^{35}(2y-1)^{28}(3y-1)^8(2y^2-4y+1) & (a_1 = \cdots = a_8 = 1), \\ 0 & (a_1 = \cdots = a_7 = 1,\ a_8 = -1). \end{cases} \tag{33}$$

Check 2 We choose random prime numbers q_i, independently from the moduli used in the modular arithmetic, and compute the resultant Eq. (29) under the substitution. Next, we compare the resultant with the circumradius formula $F_8^{(+)}(s_i;\ y)$, when $a_i := q_i$ are substituted.

In our future work, we need to improve the efficiency of the numerical interpolation method in order to compute the remaining coefficients $\tilde{P}_{10}, \ldots, \tilde{P}_0$. The difficulty lies in the increase of the matrix size; for example, 84,714 for $\tilde{P}_{11}$ but 235,516 for $\tilde{P}_0$. Since these numbers contain "unnecessary monomials," we need to find some sharper criteria for the selection of candidate monomials.

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deg in y	#terms of $\Phi_8^{(+)}$	t-deg	#terms of $F_8^{(+)}$	deg in $\sqrt{s_8}$	Successful method(s)
0	5,554,128	70	918	16	Res†
1	13,298,304	69	1,870	16	Res
2	26,940,233	68	3,432	16	Res
3	48,012,824	67	5,732	16	Res
4	77,750,132	66	8,931	16	Res
5	114,947,440	65	12,670	16	Res
6	158,302,913	64	17,129	16	Res
7	204,390,480	63	21,592	15	Res
8	250,654,676	62	26,179	15	Res
9	293,931,056	61	30,200	15	Res
10	333,471,187	60	33,748	15	Res
11	367,872,280	59	36,404	14	Res, Intp‡
12	393,876,280	58	38,662	14	Res, Intp
13	410,700,024	57	40,052	14	Res, Intp
14	418,982,117	56	41,026	14	Res, Intp
15	419,436,472	55	41,052	13	Intp
16	412,150,144	54	40,569	13	Intp
17	397,702,264	53	39,512	13	Intp
18	377,625,563	52	38,060	13	Intp
19	353,461,816	51	36,276	12	Intp
20	326,710,952	50	34,290	12	Intp
21	298,537,136	49	32,206	12	Intp
22	270,096,177	48	30,102	12	Intp
23	242,272,136	47	27,972	11	Intp
24	215,706,304	46	25,859	11	Intp
25	190,757,400	45	23,791	11	Intp
26	167,575,955	44	21,791	11	Intp
27	146,251,128	43	18,825	10	Intp
28	126,825,848	42	17,976	10	Res, Intp
29	109,294,704	41	16,183	10	Res, Intp
30	93,610,141	40	14,513	10	Res, Intp
31	79,699,496	39	12,910	9	Res, Intp
32	67,463,040	38	11,436	9	Res, Intp
33	56,784,240	37	10,026	9	Res, Intp
34	47,533,327	36	8,743	9	Res, Intp
35	39,574,496	35	7,514	8	Res, Intp
36	32,771,272	34	6,385	8	Res, Intp
37	26,990,336	33	5,260	8	Res, Intp
38	22,105,457	32	4,231	8	Res, Intp

†: Method of resultant ‡: Method of numerical interpolation

Table 1: Each coefficient in the octagon formulae $\Phi_8^{(+)}(a_i;\ y)$ and $F_8^{(+)}(s_i;\ y)$