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Some Dynamical Loci of Quartic Polynomials

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Abstract

Let $M_4(\mathbb{C})$ be all affine conjugacy classes of quartic polynomials. We define a projection Ψ_4 from $M_4(\mathbb{C})$ to $\mathbb{C}^3$ via the elementary symmetric functions of the multipliers of the fixed points. In [2], we show Ψ_4 is not surjective. The image of $M_4(\mathbb{C})$ under Ψ_4 is denoted by $\Sigma(4)$. The complement $\mathcal{E}(4) = \mathbb{C}^3 \setminus \Sigma(4)$ is called the exceptional set. On a part of the real section of $\mathcal{E}(4)$, we verify that a quartic polynomial degenerates into "twins" of quadratic polynomials. We conjecture that this phenomena holds on $\mathcal{E}(4)$.

1 Introduction

Let $\mathrm{Poly}_4(\mathbb{C})$ be the space of all quartic polynomials, and $M_4(\mathbb{C})$ be the space of all affine conjugacy classes of quartic polynomials. We define a projection Ψ_4 from $M_4(\mathbb{C})$ to $\mathbb{C}^3$ via the elementary symmetric functions of the multipliers of the fixed points. In [2], we show the projection is not surjective. The image of $M_4(\mathbb{C})$ under Ψ_4 is denoted by $\Sigma(4)$, the complement $\mathbb{C}^3 \setminus \Sigma(4)$ by $\mathcal{E}(4)$ called the exceptional set. For the cubic (resp. quadratic) polynomials, the exceptional set is empty. Unfortunately it can happen that the exceptional set $\mathcal{E}(n)$ is nonempty for $n \geqq 4$ (see [2] and [3]).

This paper consists of two parts, one is devoted to defining an algebraic variety $G(c)$, the other to analizing dynamics of $\mathrm{Poly}_4(\mathbb{C})$ on neighborhood of $\mathcal{E}(4)$.

First, we can define an algebraic variety $G(c)$, given in Section 2, that indicates essential property of the projection Ψ_4. We can derive defining equations of the exceptional set and of the branch locus from the perspective of $G(c)$. We mainly use the symbolic and algebraic computation system Risa/Asir to obtain an algebraic variety $G(c)$.

Second, we examine dynamical behavior on the parameter space $\Sigma(4)\cup\mathcal{E}(4)$. We have Theorem 16 in Section 5.

Theorem 16
There is a component $D \subset \Sigma(4)$ such that two polynomial-like maps $(U, V, p) \sim_{hb} z^2 + c$ and $(\tilde{U}, \tilde{V}, p) \sim_{hb} z^2 + \bar{c}$ are constructed for any $\langle p \rangle \in D$, and c and $\bar{c}$ converge to a common value $\tilde{c} \in \mathbb{R}$ as $\langle p \rangle \to \mathcal{E}(4)$. The limit value $\tilde{c}$ depends only on the landing point $(4, s, \frac{(s-4)^2}{4}) \in \mathcal{E}(4)$ and is written by $\tilde{c} = \frac{s-4}{8}$.

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Because the parameter values (corresponds to the "multipliers") on $\mathcal{E}(4)$ forms $\mu,\ \mu,\ 2-\mu,\ 2-\mu$, the following conjectures Conjecture 15 and Conjecture 17 seem very likely.

Conjecture 15
On the exceptional set, a quartic polynomial degenerates into "twins" of quadratic polynomials conjugate to z^2+c for some c.

Conjecture 17
None of quartic polynomial p has two disjoint quadratic-like restrictions of p such that both quadratic-like map are hybrid equivalent to a common quadratic polynomial $z^2+c,\ c\in \mathrm{M}\setminus\{\frac{1}{4}\}$, where M is Mandelbrot set.

These conjectures, given in Section 5, back with the reason why the exceptional set is not empty.

2 Definitions and Notations

Let $\mathrm{Poly}_4(\mathbb{C})$ be the space of all polynomials of the form:

$$p(z) = a_4z^4 + a_3z^3 + a_2z^3 + a_1z + a_0 \quad (a_4 \neq 0).$$

Two maps $p_1,\ p_2 \in \mathrm{Poly}_4(\mathbb{C})$ are *holomorphically conjugate*, denoted by $p_1 \sim p_2$, if and only if there exists $g \in \mathfrak{A}(\mathbb{C})$ with $g\circ p_1 \circ g^{-1} = p_2$, where $\mathfrak{A}(\mathbb{C})$ is the group of all affine transformations.

The space, $\mathrm{Poly}_4(\mathbb{C})/_\sim$, of holomorphic conjugacy classes $\langle p\rangle$ of quartic polynomials is denoted by $\mathrm{M}_4(\mathbb{C})$.

For each $p(z) \in \mathrm{Poly}_4(\mathbb{C})$, let $z_1,\ \cdots,\ z_4,\ z_5 = \infty$ be the fixed points of p, and $\mu_1,\ \cdots,\ \mu_4,\ \mu_5 = 0$ the multipliers of z_i (i.e. $\mu_i = p'(z_i)$). Let $\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_5$ be the elementary symmetric functions of these multipliers

$$\begin{aligned}
\sigma_1 &= \mu_1+\mu_2+\mu_3+\mu_4,\\
\sigma_2 &= \mu_1\mu_2+\mu_1\mu_3+\mu_1\mu_4+\mu_2\mu_3+\mu_2\mu_4+\mu_3\mu_4,\\
\sigma_3 &= \mu_1\mu_2\mu_3+\mu_1\mu_2\mu_4+\mu_1\mu_3\mu_4+\mu_2\mu_3\mu_4,\\
\sigma_4 &= \mu_1\mu_2\mu_3\mu_4,\\
\sigma_5 &= 0.
\end{aligned}$$

These multipliers are *invariant* under the action of (conjugation) $\mathfrak{A}(\mathbb{C})$.

The holomorphic index of a rational function f at a fixed point $\zeta \in \mathbb{C}$ is defined to be the complex number

$$\iota(f,\ \zeta) = \frac{1}{2\pi i}\oint \frac{dz}{z-f(z)},$$

where we integrate in a small loop in the positive direction around ζ.

The following results are well known as "Fatou's index theorem":

- If ζ is a fixed point of multiplier $\mu \neq 1$, then $\iota(f,\ \zeta) = \frac{1}{1-\mu}$.
- For any polynomial p which is not the identity map,

$$\sum_{\zeta\in\mathbb{C}} \iota(p,\ \zeta) = 0, \tag{1}$$

  where this summation is over all fixed points of p.

By an automorphism of a polynomial map p we will mean an affine transformation g that commutes with p. The collection $\mathrm{Aut}(p)$ of all automorphisms of p forms a finite group. The symmetry locus of the polynomials of degree n is defined to be the set $\mathcal{S}_n$ ($\subset \mathrm{M}_n$) consisting of all conjugacy classes $\langle p \rangle$ of polynomial maps admitting non-trivial automorphisms.

A *polynomial-like* map of degree d is a triple $(U,\ V,\ f)$ where U and V are topological disks, with V relatively compact in U, and $f\ :\ V \to U$ is analytic, proper of degree d.

An orientation-preserving homeomorphism f of a domain D onto another, D', is called a *quasiconformal map* if f is ACL on every rectangle

$$R = \{z = x + iy \,|\, a \leqq x \leqq b,\ c \leqq y \leqq d\},$$

i.e.,

- $f(x + iy)$ is absolutely continuous on $[a,\ b]$ with respect to x for almost every fixed y,
- $f(x + iy)$ is absolutely continuous on $[c,\ d]$ with respect to y for almost every fixed x, and
- there is a constant $k < 1$ such that $|f_{\bar{z}}| \leqq k|f_z|$ almost everywhere on D.

The filled-in Julia set K_f of a polynomial-like map $(U,\ V,\ f)$ is defined by

$$K_f = \bigcap_{n \geqq 0} f^{-n}(V).$$

Polynomial-like maps $(U,\ V,\ f)$ and $(\widetilde{U},\ \widetilde{V},\ \tilde{f})$ are *hybrid equivalent* $f \sim_{hb} \tilde{f}$, if there exists a quasi-conformal map h from a neighborhood of K_f to a neighborhood of $K_{\tilde{f}}$, such that $h \circ f = \tilde{f} \circ h$ near K_f and $\bar{\partial} h = 0$ almost everywhere on K_f.

From Straightening Theorem in [1], every polynomial-like map $(U,\ V,\ f)$ of degree d is hybrid equivalent to a polynomial P of degree d. If K_f is connected then P is unique up to conjugation by an affine map.

3 The projection from $\mathrm{M}_4(\mathbb{C})$ to $\mathbb{C}^3$

Let $\sigma_1,\ \cdots,\ \sigma_4$ be the elementary symmetric functions of the multipliers of $p(z) \in \mathrm{Poly}_4(\mathbb{C})$. The following relation holds by Fatou's index theorem.

Lemma 1 (Theorem 1 in [2])
Among $\sigma_i{}'s$, there is a linear relation

$$4 - 3\sigma_1 + 2\sigma_2 - \sigma_3 = 0.$$

For a monic and centered quartic polynomial $z^4 + c_2 z^2 + c_1 z + c_0$, the three values $\sigma_1,\ \sigma_2,\ \sigma_4$ are determined by *Transformation formula*:

$$\begin{aligned}
\sigma_1 &= -8c_1 + 12,\\
\sigma_2 &= 4c_2^3 - 16c_0c_2 + 18c_1^2 - 60c_1 + 48,\\
\sigma_4 &= 16c_0c_2^4 + (-4c_1^2 + 8c_1)c_2^3 - 128c_0^2c_2^2 + (144c_0c_1^2 - 288c_0c_1 + 128c_0)c_2\\
&\qquad -27c_1^4 + 108c_1^3 - 144c_1^2 + 64c_1 + 256c_0^3.
\end{aligned}$$

```
Risa/Asir commands for getting Transformation formula:
  P=z^4+c2*z^2+c1*z+c0;
  Fix1=z0+z1+z2+z3;
  Fix2=z0*(z1+z2+z3)+z1*(z2+z3)+z2*z3-c2;
  Fix3=z0*z1*z2+z0*z1*z3+z0*z2*z3+z1*z2*z3+c1-1;
  Fix4=z0*z1*z2*z3-c0;
  DP=diff(P,z);
  M0=subst(DP,z,z0)-m0;
  M1=subst(DP,z,z1)-m1;
  M2=subst(DP,z,z2)-m2;
  M3=subst(DP,z,z3)-m3;
  S1=m0+m1+m2+m3-s1 ;
  S2=m0*(m1+m2+m3)+m1*(m2+m3)+m2*m3-s2;
  S4=m0*m1*m2*m3-s4;
  load("gr");
  G=gr([Fix1,Fix2,Fix3,Fix4,M0,M1,M2,M3,S1,S2,S4],
       [m0,m1,m2,m3, z0,z1,z2,z3, s4,s2,s1, c2,c1,c0],
       [[0,4],[0,4],[0,3],[0,3]]);
```

There is a natural projection

$$\begin{array}{cccc} \Psi_4 : & \mathrm{M}_4(\mathbb{C}) & \longrightarrow & \Sigma(4) \\ & \Cup & & \Cup \\ & \langle p \rangle & \longmapsto & (\sigma_1,\ \sigma_2,\ \sigma_4), \end{array}$$

where $\Sigma(4)$ is the image of $\mathrm{M}_4(\mathbb{C})$ under Ψ_4. The complement $\mathbb{C}^3 \setminus \Sigma(4)$ is denoted by $\mathcal{E}(4)$, and called the *exceptional set*.

Under the conjugacy of the action of $\mathfrak{A}(\mathbb{C})$, it can be assumed that any quartic polynomial is "monic" and "centered":

$$p(z) = z^4 + c_2 z^2 + c_1 z + c_0.$$

There are three monic and centered polynomials in any conjugacy class $\langle p \rangle$ except for $\langle z^4 + z \rangle$, and they are transformed each other under the action of $G_3 = \{1,\ \omega,\ \omega^2\}$, where ω is third roots of unity.

After following procedures 1 – 4, we obtain a parametrized algebraic variety. This variety indicates essential property of the projection Ψ_4.

1. for a point $\langle p \rangle \in \mathrm{M}_4(\mathbb{C})$, choose a monic and centered representative $z^4 + c_2 z^2 + c_1 z + c_0$,
2. getting rid of the affine ambiguity on "Transformation formula", set $c := c_2^3$ (if $c_2 = 0$, set $\tilde{c} := c_0^3$), and
3. rebuild Transformation formula of $\sigma_1,\ \sigma_2,\ \sigma_4,\ c,\ c_0,\ c_1$ variables,
4. remove two variables $c_0,\ c_1$, from the above formula.

Definition 2
We define an algebraic variety in $\mathbb{C}^3$ with a parameter $c \in \mathbb{C}$,

$$G(c): \quad 262144(\sigma_1-4)^2c^2 + 1024(27\sigma_1^4 + (-144\sigma_2 - 576)\sigma_1^2 + (384\sigma_2 + 1280)\sigma_1 + 128\sigma_2^2 \\ -256\sigma_2 - 512\sigma_4 - 768)c + (9\sigma_1^2 + 24\sigma_1 - 32\sigma_2 - 48)^3 = 0.$$

```
Risa/Asir commands for getting G(c) :
  Sgm1=-8*c1+12-s1;
  Sgm2=4*c2^3-16*c0*c2+18*c1^2-60*c1+48-s2;
  Sgm4=16*c0*c2^4+(-4*c1^2+8*c1)*c2^3-128*c0^2*c2^2+(144*c0*c1^2
       -288*c0*c1+128*c0)*c2-27*c1^4+108*c1^3-144*c1^2+64*c1
       +256*c0^3-s4;
  CC=c2^3-c;
  load("gr");
  G1=gr([Sgm1,Sgm2,Sgm4,CC],
        [c0,c1,c2, c, s1,s2,s4],[[0,3],[0,1],[0,3]]);
```

The number of conjugacy classes corresponding to a point $(\sigma_1,\ \sigma_2,\ \sigma_4) \in \mathbb{C}^3$ is equal to the number of allowable parameter values c on $G(c)$. Namely, we have the following theorem by counting the number of solution c of the defining equation of $G(c)$.

Theorem 3
For $(\sigma_1,\ \sigma_2,\ \sigma_4) \in \mathbb{C}^3$, number of the elements of set $\Psi_4^{-1}(\sigma_1,\ \sigma_2,\ \sigma_4)$ are ∞, 0, 1 or 2:

Case 1 $\#\Psi_4^{-1}(\sigma_1,\ \sigma_2,\ \sigma_4) = \infty$ if and only if $(\sigma_1,\ \sigma_2,\ \sigma_4) = (4,\ 6,\ 1)$. And further, we can precisely formulate,

$$\Psi_4^{-1}(4,\ 6,\ 1) = \left\{p_a(z) = (z^2-a)^2 + z\right\}_{a\in\mathbb{C}} \quad (\text{note } p_a \sim p_{\pm\omega a}).$$

Case 2 $\#\Psi_4^{-1}(\sigma_1,\ \sigma_2,\ \sigma_4) = 0$ if and only if the point $(\sigma_1,\ \sigma_2,\ \sigma_4)$ cannot belong to $G(c)$ for any c. And further, we can give defining equation of the exceptional set $\mathcal{E}(4)$,

$$(\sigma_1,\ \sigma_2,\ \sigma_4) = \left(4,\ s,\ \frac{(s-4)^2}{4}\right), \quad s \neq 6. \tag{2}$$

Case 3 $\#\Psi_4^{-1}(\sigma_1,\ \sigma_2,\ \sigma_4) = 1$ if and only if discriminant of the defining equation of $G(c)$ vanishes or $\sigma_1 = 4$.

Case 4 $\#\Psi_4^{-1}(\sigma_1,\ \sigma_2,\ \sigma_4) = 2$, for the remains of the above.

Theorem 3 leads immediately to the following two corollaries which are justified for structural and topological reasons.

Corollary 4
The exceptional set $\mathcal{E}(4)$ is contained in the plane $\{(4,\ \sigma_2,\ \sigma_4)\} \cong \mathbb{C}^2$.

Corollary 5
None of quartic polynomial has the fixed points of the multipliers $\mu,\ \mu,\ 2-\mu,\ 2-\mu,\ (\mu \neq 1)$.

Corollary 5 is clear from (2) and the relation between roots and coefficients.

```
Risa/Asir commands for getting the multipliers:
  S1=m0+m1+m2+m3-4 ;
  S2=m0*(m1+m2+m3)+m1*(m2+m3)+m2*m3-s;
  S4=m0*m1*m2*m3-(s-4)^2/4;
  Index=nm(1/(1-m0)+1/(1-m1)+1/(1-m2)+1/(1-m3));
  load("gr");
  G2=gr([S1,S2,S4,Index],[s,m3,m2,m1,m0],2);
```

Remark 6
The discriminant of the defining equation of $G(c)$ is given as follows:

$$\begin{aligned}\mathrm{Discr} \;=\; & 1073741824(54\sigma_1^5+(-81\sigma_2-27\sigma_4-135)\sigma_1^4+(36\sigma_2^2-144\sigma_2-1008)\sigma_1^3\\ & +(-4\sigma_2^3+360\sigma_2^2+(144\sigma_4+2976)\sigma_2+576\sigma_4+4192)\sigma_1^2\\ & +(-160\sigma_2^3-2176\sigma_2^2+(-384\sigma_4-6400)\sigma_2-1280\sigma_4-5376)\sigma_1+16\sigma_2^4+448\sigma_2^3\\ & +(-128\sigma_4+2176)\sigma_2^2+(256\sigma_4+3840)\sigma_2+256\sigma_4^2+768\sigma_4+2304).\end{aligned}$$

Therefore $Discr = 0$ is a surface in $\mathbb{C}^3$.

Theorem 7
The symmetry locus can be formulated as follows:

$$\begin{cases}\sigma_1 = s,\\ \sigma_2 = 3(3s-4)(s+4)/32, \qquad (s \in \mathbb{C}).\\ \sigma_4 = -(3s-4)^3(s-12)/4096.\end{cases} \tag{3}$$

Proof The symmetry locus is the singular part of the surface defined by $Discr = 0$ (see [2]).

$$Discr = Discr_{s_1} = Discr_{s_2} = Discr_{s_4} = 0, \qquad \sigma_1 \neq 4.$$

Solving these equations we have (3). ∎

Remark 8
The point $(4,\ 6,\ 1)$ is the unique intersection point of the symmetry locus and the plane $\sigma_1 = 4$.

4 Loci $\mathrm{Per}_1(\mu)$ on the space $\{(4,\ s_2,\ s_4)\}$

In this section, we consider dynamical behavior on the real section $\mathbb{R}^2 \cong \{(4,\ s_2,\ s_4)\}$.

The locus $\mathrm{Per}_1(\mu)$ be the set of all conjugacy classes $\langle p\rangle$ of maps p having a fixed point of multiplier μ.

Proposition 9

For each $\mu \in \mathbb{C}$, $\mathrm{Per}_1(\mu)$ is a straight line with the following defining equation:

$$\mathrm{Per}_1(\mu):\ \sigma_4 - (2\mu - \mu^2)\sigma_2 + \mu^4 - 4\mu^3 + 8\mu = 0.$$

Proof The multipliers at the fixed points are the roots of the equation,

$$\mu^4 - \sigma_1\mu^3 + \sigma_2\mu^2 - \sigma_3\mu + \sigma_4 = 0.$$

From the linear relation of Lemma 1, we have the defining equation of $\mathrm{Per}_1(\mu)$. ∎

We remark that the multipliers of a quartic polynomial on the real plane $\{(4,\ \sigma_2,\ \sigma_4)\}$ are 'four real values', 'two real and a pair of complex conjugates', or 'two pair of complex conjugates'.

4.1 $\mathrm{Per}_1(\mu)$ ($\mu \in \mathbb{R}$)

At first we consider $\mu \in \mathbb{R}$. In this case we can illustrate the figure of $\mathrm{Per}_1(\mu)$. (See Figure 1.) The following results are easily verified.

Proposition 10

For $\langle p\rangle \in \{(4,\ \sigma_2,\ \sigma_4)\} \cap \Sigma(4)$, the corresponding multipliers of p are $\mu,\ 2-\mu,\ \lambda,\ 2-\lambda$.

Proof It is clear from the relation between roots and coefficients. ∎

```
Risa/Asir commands for getting the multipliers:
  S1=m0+m1+m2+m3-4 ;
  S2=m0*(m1+m2+m3)+m1*(m2+m3)+m2*m3-s2;
  S4=m0*m1*m2*m3-s4;
  Index=nm(1/(1-m0)+1/(1-m1)+1/(1-m2)+1/(1-m3));
  load("gr");
  G3=gr([S1,S2,S4,Index],[s2,s4,m3,m2,m1,m0],2);
```

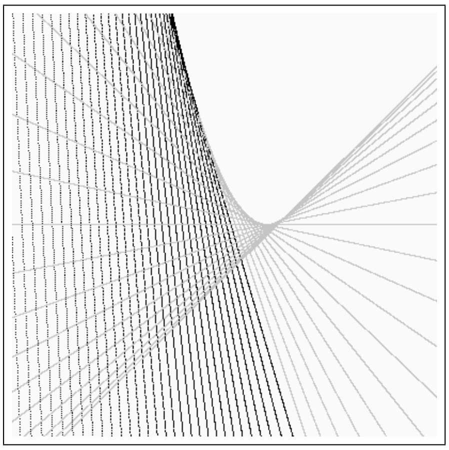

Figure 1:

The left figure shows $\mathrm{Per}_1(\mu)$ $(-10 < \mu < 1)$:
$-20 < s_2,\ s_4 < 20$,
Gray lines mean $\mathrm{Per}_1(\mu)$ $(|\mu| \geqq 1)$ and
black lines mean $\mathrm{Per}_1(\mu)$ $(|\mu| < 1)$.

Corollary 11

- If p has a attracting fixed point then p has a repelling fixed point with positive multiplier.
- If p has a repelling fixed point with negative multiplier then p has a repelling fixed point with positive multiplier.

  Namely, each line of Figure 1 is overlapped by a line $\mathrm{Per}_1(\mu)$ for some $\mu > 1$, and p cannot have three attracting fixed points.

4.2 $\mathrm{Per}_1(\mu)$ and $\mathrm{Per}_1(\bar{\mu})$

Next, we consider the multipliers of a quartic polynomial are 'two real and a pair of complex conjugates'. In this case, the multipliers are $1 \pm i\beta$, λ, and $2-\lambda$ from Proposition 10. Then we have the following from Proposition 9.

Proposition 12
For each $\beta \in \mathbb{R}$, $\mathrm{Per}_1(1 \pm i\beta)$ is a straight line with the following defining equation:

$$\mathrm{Per}_1(1 \pm i\beta): \quad \sigma_4 = (1+\beta^2)\sigma_2 - (1+\beta^2)(5+\beta^2).$$

Proof Removing λ from two equations $\sigma_2 = 5+\beta^2+\lambda(2-\lambda)$ and $\sigma_4 = (1+\beta^2)\lambda(2-\lambda)$, we have the above defining equation of $\mathrm{Per}_1(1 \pm i\beta)$. ∎

Note that these loci are corresponds to repelling fixed points.

Now, we consider the last case: multipliers of a quartic polynomial are 'two pair of complex conjugates'. In this case, the multipliers are $a \pm ib$ and $2-a \pm ib$ from Proposition 10. Because defining equation of $\mathrm{Per}_1(\mu)$ can express a line on the real plane no longer, we need a new device $\widetilde{\mathrm{Per}_1}(\tilde{\mu})$ for illustrating figures of $\mathrm{Per}_1(\mu)$. (See Figure 2.)

The locus $\widetilde{\mathrm{Per}_1}(\tilde{\mu})$ be the set of all conjugacy classes $\langle p \rangle$ of maps p having a fixed point of multiplier μ with $\tilde{\mu} = \mu\bar{\mu}$.

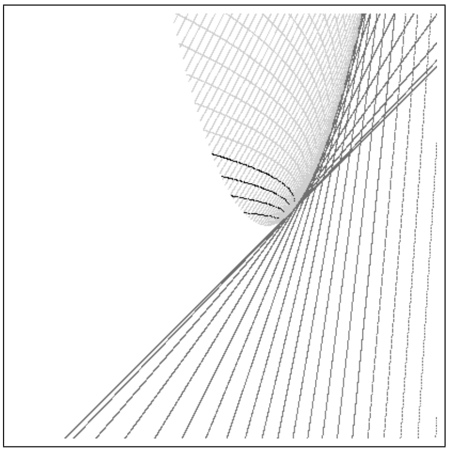

Figure 2:

The left figure shows $\mathrm{Per}_1(1 \pm i\beta)$ and $\widetilde{\mathrm{Per}_1}(\tilde{\mu})$.
$-20 < s_2,\ s_4 < 20$,
Dark gray lines mean $\mathrm{Per}_1(1 \pm i\beta)$,
gray curves mean $\widetilde{\mathrm{Per}_1}(\tilde{\mu}),\ t \geqq 1$ and
black curves mean $\widetilde{\mathrm{Per}_1}(\tilde{\mu}),\ t < 1$.

Proposition 13
In the case that the multipliers are $a \pm ib$ and $2-a \pm ib$, we have a defining equation of $\widetilde{\mathrm{Per}_1}(\tilde{\mu})$.

$\widetilde{\mathrm{Per}_1}(\tilde{\mu})$:

$$\sigma_4^2 - 2(\tilde{\mu}^2+2\tilde{\mu})\sigma_4 + \tilde{\mu}^4 - 4\tilde{\mu}^3 + (\sigma_2 - 16)\tilde{\mu}^2 = 0,$$

where $\tilde{\mu} = a^2 + b^2$.

Proof In this case the multipliers are $a \pm ib$ and $2-a \pm ib$. By setting $\tilde{\mu} = a^2+b^2$ for two equations $\sigma_2 = -2a^2+4a+4+2b^2$ and $\sigma_4 = (a^2+b^2)((2-a)^2+b^2)$, we have

$$\sigma_2 = -4a^2+4a+4+2\tilde{\mu}, \qquad \sigma_4 = \tilde{\mu}(\tilde{\mu}-4a+4). \tag{4}$$

Removing a from the above two equations, we have a defining equation of $\widetilde{\mathrm{Per}_1}(\tilde{\mu})$. ∎

Remark 14
If $0 \leqq t < 1$, $\widetilde{\mathrm{Per}_1}(\tilde{\mu})$ corresponds to polynomials having two attracting fixed points of multiplier $a+ib$ and $a-ib$. As $a,\ b \in \mathbb{R}$, the discriminant $4+4(4+2\tilde{\mu}-\sigma_2)$ of the left of (4) must be positive. Therefore, on a region $\{(4,\ \sigma_2,\ \sigma_4)\,|\,\sigma_2 < -\frac{1}{4}(\sigma_4^2 - 6\sigma_4 - 19),\ \sigma_4 < \frac{(2-\sigma_2)^2}{4}\}$, corresponding polynomial p have two attracting fixed points of multipliers $a \pm ib$.

5 Dynamics on the exceptional set

The lines $\{\mathrm{Per}_1(\mu)\}$ have a close relation with the exceptional set. As an example, we give the following results directly obtained by the results in the section 4.1 and 4.2.

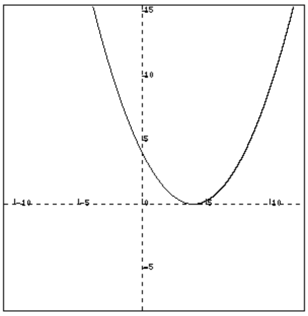

Figure 3:

The left figure shows the real section of the exceptional set

$$\mathcal{E}(4) : \left(4,\ s,\ \frac{(s-4)^2}{4}\right), \quad (s \neq 6).$$

- On the plane $\{(4,\ s_2,\ s_4)\} \cong \mathbb{R}^2$, the envelopes of the lines $\{\mathrm{Per}_1(\mu)\}_{\mu\in\mathbb{R}}$ and of $\{\mathrm{Per}_1(1 \pm i\beta)\}_{\beta\in\mathbb{R}}$ coincides with the exceptional set. (See Figure 1, 2 and 3.)
- On the region $\left\{ (4,\ \sigma_2,\ \sigma_4) \mid \sigma_4 < \frac{(2-\sigma_2)^2}{4} \right\}$ that bounded by the exceptional set, corresponding quartic polynomial has the fixed points of the multiplier with two pair of complex conjugates.

Conjecture 15
On the exceptional set, a quartic polynomial degenerates into "twins" of quadratic polynomials conjugate to $z^2 + c$ for some c.

Theorem 16
There is a component $D \subset \Sigma(4)$ such that two polynomial-like maps $(U,\ V,\ p) \sim_{hb} z^2 + c$ and $(\tilde{U},\ \tilde{V},\ p) \sim_{hb} z^2 + \bar{c}$ are constructed for any $\langle p\rangle \in D$, and c and $\bar{c}$ converge to a common value $\tilde{c} \in \mathbb{R}$ as $\langle p\rangle \to \mathcal{E}(4)$. The limit value $\tilde{c}$ depends only on the landing point $(4,\ s,\ \frac{(s-4)^2}{4}) \in \mathcal{E}(4)$ and is written by $\tilde{c} = \frac{s-4}{8}$.

Proof On a region $\{(4,\ \sigma_2,\ \sigma_4) \mid \sigma_2 < -\frac{1}{4}(\sigma_4^2 - 6\sigma_4 - 19),\ \sigma_4 < \frac{(2-\sigma_2)^2}{4}\}$, any corresponding polynomial $p(z)$ has two attracting fixed points of multiplier $\mu,\ \overline{\mu}$. Dynamics of $p(z)$ are symmetry for the real axis. (See Figure 4.) Therefore we can choose suitable topological disk $U,\ \widetilde{U}$ bounded by equipotential curves such that $(U,\ V,\ p)$ and $(\widetilde{U},\ \widetilde{V},\ p)$ $(U \cap \widetilde{U} = \emptyset)$ are quadratic-like maps hybrid equivalent to $z^2 + c$ and $z^2 + \overline{c}$ respectively.

Then, if $\langle p\rangle$ converges to a point $(4,\ s,\ \frac{(s-4)^2}{4}) \in \mathcal{E}(4)$, two parameters c and $\overline{c}$ are converges to common value $\frac{s-4}{8} \in \mathbb{R}$. (See Figure 6 and 7. Figure 8–11 show another example.) ∎

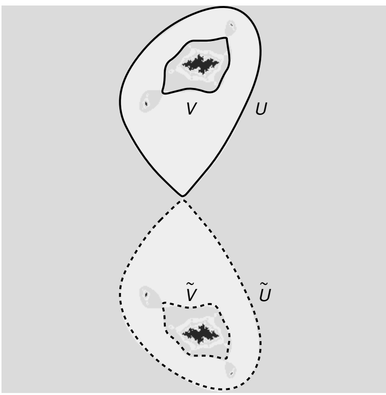

Figure 4: $(4,\ -1.7696160,\ 8.8480801)$, Julia set of $p(z) = z^4 + 3.8199z^2 + z + 3.775218$, $-2 < \Re z,\ \Im z < 2$

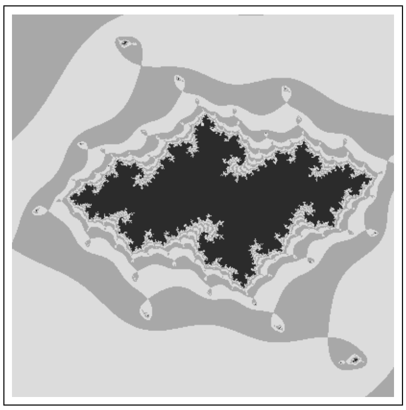

Figure 5: Julia set of $p(z) = z^4 + 3.8199z^2 + z + 3.775218$, $-0.2 < \Re z < 0.28$, $1.137 < \Im z < 1.617$

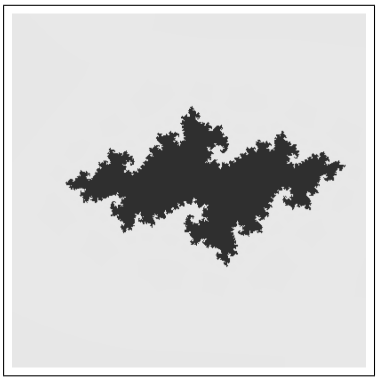

Figure 6: Julia set of quadratic - like map $-0.2 < \Re z < 0.28$, $1.137 < \Im z < 1.617$

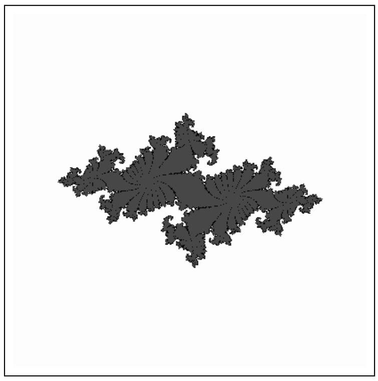

Figure 7: Julia set of $p_c(z) = z^2 + (-0.726 + 0.183i)$, $-2 < \Re z, \ \Im z < 2$.

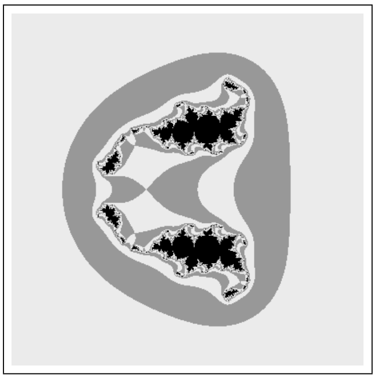

Figure 8: (4, 3, 7), Julia set of $p(z) = z^4 + 0.62996z^2 + z + 0.39685$, $-2 < \Re z, \ \Im z < 2$

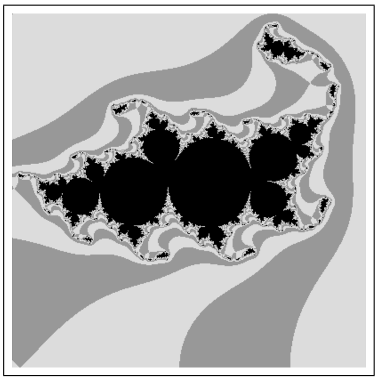

Figure 9: Julia set of $p(z) = z^4 + 0.62996z^2 + z + 0.39685$, $-0.5 < \Re z < 0.8, \ 0 < \Im z < 1.3$

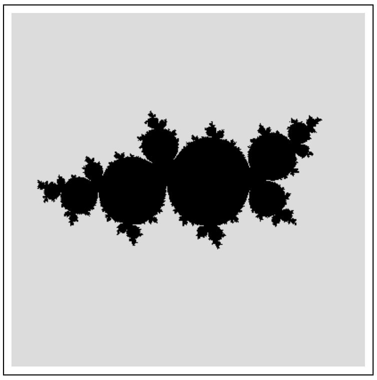

Figure 10: Julia set of quadratic-like map $-0.5 < \Re z < 0.8, \ 0 < \Im z < 1.3$

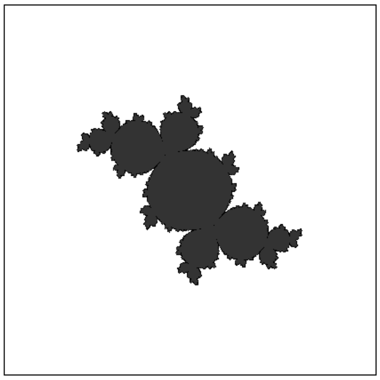

Figure 11: Julia set of $p_c(z) = z^2 + (-0.125 + 0.65i)$, $-2 < \Re z, \ \Im z < 2$.

6 Dynamics on the point (4, 6, 1) $\in \Sigma(4)$

One parameter family $\left\{p_a(z) = (z^2 - a)^2 + a\right\}_{a \in \mathbb{C}}$ $(p_a \sim p_{\pm\omega a})$ corresponds to the point (4, 6, 1). There are maps p in this family such that p have two disjoint quadratic-like restriction, hybrid equivalent to common quadratic map $z^2 + \frac{1}{4}$. (See Figure 12 and 13.)

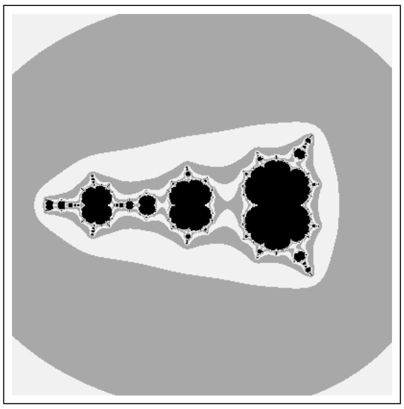

Figure 12: Julia set of $p(z) = z^4 - 2z^2 + z + 1$, $-2 < \Re z,\ \Im z < 2$. (4, 6, 1) $\in \Sigma(4)$

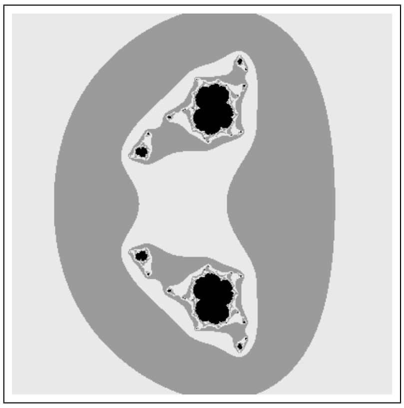

Figure 13: Julia set of $p(z) = z^4 + 2z^2 + z + 1$, $-2 < \Re z,\ \Im z < 2$. (4, 6, 1) $\in \Sigma(4)$

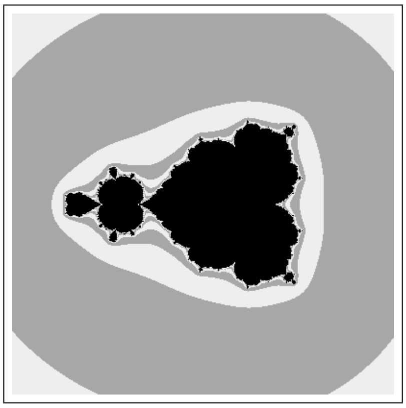

Figure 14: Julia set of $p(z) = z^4 - z^2 + z + 0.25$, $-2 < \Re z,\ \Im z < 2$. (4, 6, 1) $\in \Sigma(4)$

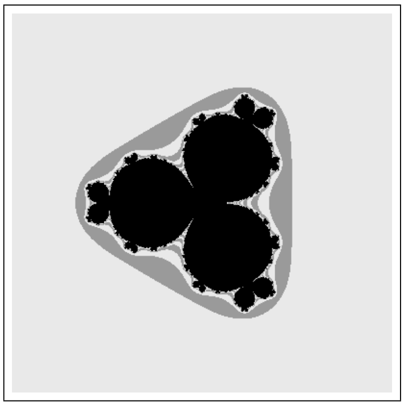

Figure 15: Julia set of $p(z) = z^4 + z$, $-2 < \Re z,\ \Im z < 2$. (4, 6, 1) $\in \Sigma(4)$

On the other hand, in Figure 14, the largest Fatou components contains two critical points. Therefore in this case p cannot have two disjoint quadratic-like restriction. The quartic polynomial in Figure 15 has unique parabolic fixed point at the origin.

Conjecture 17

None of quartic polynomial p have two disjoint quadratic-like restrictions of p such that both quadratic-like map are hybrid equivalent to a common quadratic polynomial $z^2 + c,\ c \in \mathrm{M} \setminus \{\frac{1}{4}\}$, where M is Mandelbrot set.

This conjecture gives a reason why the exceptional set is not empty.

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