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An Application of Gröbner Bases for the Moduli of Hypersurface Simple $K3$ Singularities

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Abstract

For many classes of singularities there exist normal forms. We know the weights of hypersurface simple $K3$ singularities by nondegenerate polynomials and obtained examples. In this paper, we try to decide the non-degeneracy conditions of unimodular, bimodular and trimodular type simple $K3$ singularities. We show that they are obtained by using Gröbner bases.

1 Introduction

Let $f_1,\ f_2,\ \ldots,\ f_r$ be holomorphic functions defined in an open set U of the complex space $\mathbf{C^n}$. Let X be the analytic set $f_1^{-1}(0) \cap \ldots \cap f_r^{-1}(0)$. Let $x \in X$, and let $g_1,\ g_2,\ \ldots,\ g_s$ be a system of generators of ideal $I(X)_{x_0}$ of the holomorphic functions which vanish identically on a neighborhood of x_0 in X. x_0 is called a simple point of X if the matrix $(\partial g_i/\partial x_j)$ attains its maximal rank. Otherwise, x_0 is called a singlar point(singularity) of X. (For $r = 1,\ x_0$ is called a hypersurface singularity of X.)

Let V be an analytic set in $\mathbf{C^n}$. A singular point x_0 of V is said to be isolated if, for some open neighborhood W of x_0 in $\mathbf{C^n}$, $W \cap V - \{x_0\}$ is a smooth submanifold of $W - \{x_0\}$.

Let $(X,\ x)$ be a germ of normal isolated singularity of dimension n. Suppose that X is a Stein space. Let $\pi : (M, E) \rightarrow (X, x)$ be a resolution of singularity. Then for $1 \leq i \leq n-1$, $dim(R^i\pi, \vartheta_M)_X$ is finite. $R^i\pi, \vartheta_M$ has support on x. They are independent of the resolution.
In fact

$$dim(R^i\pi, \vartheta_M)_X = dim H_X^{i+1}(X, \vartheta_M) \quad (1 \leq i \leq n-2)$$

and

$$dim(R^{n-1}\pi, \vartheta_M)_X = \frac{dim\Gamma(X - \{x\}, \vartheta K)}{L^2(X - \{x\})}.$$

where $L^2(X - \{x\})$ is the subspace of $\Gamma(X - \{x\}, \vartheta K)$ consisting of n-form on $X - \{x\}$ which are square integrable near x.

We denote them by

$$h^i(X,\ x) := dim(R^i\pi, \vartheta_M)_X \quad (1 \leq i \leq n-2)$$

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and

$$P_g(X,\ x)\ :=\ dim(R^i\pi, \vartheta_M)_X.$$

The invariant $P_g(X,\ x)$ is called the geometric genus of $(X,\ x)$.

In the theory of two-dimensional singularities, simple elliptic singularities and cusp singularities are regarded as the next most reasonable class of singularities after rational singularities. Cusp singularities appear on the Satake compactifications of Hilbert modular surfaces and have loops of rational curves as the exceptional sets of the minimal resolution. Simple elliptic singularities were investigated by Saito([3]) in detail. By definition, each of them has a nonsingular elliptic curve as the exceptional set of the minimal resolution. Here we are interested especially in a hypersurface *simple elliptic* singularity $(X,\ x)$. In this case, the defining equation of $(X,\ x)$ is given by one of the following in some coordinates $z_1,\ z_2,\ z_3$ around x,

$$\begin{array}{lcl} \tilde{E}_6 & : & z_1^3 + z_2^3 + z_3^3 + \lambda_1 z_1 z_2 z_3 \ = \ 0 \quad (E^2 \ = \ -3) \\ \tilde{E}_7 & : & z_1^2 + z_2^4 + z_3^4 + \lambda_2 z_1 z_2 z_3 \ = \ 0 \quad (E^2 \ = \ -2) \\ \tilde{E}_8 & : & z_1^2 + z_2^3 + z_3^6 + \lambda_3 z_1 z_2 z_3 \ = \ 0 \quad (E^2 \ = \ -1) \end{array}$$

with the parameter satisfying $\lambda_1^3 + 27 \neq 0,\ \lambda_2^4 - 64 \neq 0,\ \lambda_3^6 - 432 \neq 0$ and corresponding to the moduli of the elliptic curve E which appears as the exceptional set.

What are natural generalizations in three-dimensional case of those singularities? They are purely elliptic singularities. And we regard simple $K3$ singularities as natural generalizations of simple elliptic singularities in three-dimensional case. We define the simple $K3$ singularities.

2 Simple $K3$ singularities

The notion of a simple $K3$ singularity was defined by Ishii and Watanabe [4] as a three-dimensional Gorenstein purely elliptic singularity of $(0,2)$-type, whereas a simple elliptic singularity is two-dimensional purely elliptic singularity of $(0,1)$-type.

Definition 1 ([6])
Let $(X,\ x)$ be a normal isolated singularity. For any positive integer m,

$$\delta_m(X,\ x)\ =\ \frac{dim_c\Gamma(X-\{x\}, \vartheta(mK))}{L^{2/m}(X-\{x\})},$$

where K is the canonical line bundle on $X - \{x\}$, and $L^{2/m}(X - \{x\})$ is the set of all $L^{2/m}$-integrable (at x) holomorphic m-tuple n-forms on $X - \{x\}$.

Then δ_m is finite and does not depend on the choice of a Stein neighborhood on X.

Definition 2 ([6])
A singularity $(X,\ x)$ is said to be purely elliptic if $\delta_m\ =\ 1$ for every positive integer m.

When X is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities, i.e., there exists a non-vanishing holomorphic 2-form on $X - \{x\}$.

Definition 3 ([4])
A three-dimensional singularity $(X,\ x)$ is a simple $K3$ singularity if the following two equivalent conditions are satisfied:

(1) $(X,\ x)$ is Gorenstein purely elliptic of $(0,2)$-type.

(2) $(X,\ x)$ is quasi-Gorenstein and the exceptional divisor E is a normal K3 surface for any minimal resolution $\pi\ :\ (\tilde{X},\ E)\ \rightarrow\ (X,\ x)$.

Simple elliptic singularities and cusp singularities are characterized as two-dimensional purely elliptic singularities of $(0,1)$-type and of $(0,0)$-type, respectively. The notion of a simple $K3$ singularity is defined as a three-dimensional isolated Gorenstein purely elliptic singularity of $(0,2)$-type.

Let $f\ \in\ \mathbf{C}[\mathrm{z}_0,\ \mathrm{z}_1,\ \mathrm{z}_2,\ \mathrm{z}_3]$ be a polynomial which is nondegenerate with respect to its Newton boundary $\Gamma(f)$ in the sense of [5], and whose zero locus $X\ =\ \{f\ =\ 0\}$ in $\mathbf{C^4}$ has an isolated singularity at the origin $0\ \in\ \mathbf{C^4}$.Then the condition for $(X,\ 0)$ to be a simple $K3$ singularity is given by a property of the Newton boundary $\Gamma(f)$ of f.

Next we consider the case where $(X,\ x)$ is a hypersurface singularity defined by a nondegenerate polynomial $f\ =\ \sum a_\nu z^\nu\ \in\ \mathbf{C}[\mathrm{z}_0,\ \mathrm{z}_1,\ \ldots,\ \mathrm{z}_\mathrm{n}]$, and $x\ =\ 0\ \in\ \mathbf{C^{n+1}}$. We denote by $\mathbf{R_0}$ the set of all nonnegative real numbers. Recall that the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of $\bigcup_{a_\nu\neq 0}(\nu\ +\ \mathbf{R_0^{n+1}})$ in $\mathbf{R^{n+1}}$.

For any face Δ of $\Gamma_+(f)$, set $f_\Delta\ :=\ \sum_{\nu\in\Delta} a_\nu z^\nu$. We say f to be nondegenerate, if

$$\frac{\partial f_\Delta}{\partial z_0}\ =\ \frac{\partial f_\Delta}{\partial z_1}\ =\ \ldots\ =\ \frac{\partial f_\Delta}{\partial z_n}\ =\ 0$$

has no solution in $(\mathbf{C}^*)^{\mathbf{n+1}}$ for any face Δ.

When f is nondegenerate, the condition for $(X,\ x)$ to be a purely elliptic singularity is given as follows:

Theorem 4 ([7])
Let f be a nondegenerate polynomial and suppose $X\ =\ \{f\ =\ 0\}$ has an isolated singularity at $x\ =\ 0\ \in\ \mathbf{C^{n+1}}$.

(1) $(X,\ x)$ is purely elliptic if and only if $(1,\ 1,\ \ldots,\ 1)\ \in\ \Gamma(f)$.

(2) Let $n\ =\ 3$ and let Δ_0 be the face of $\Gamma(f)$ containing $(1,\ 1,\ 1,\ 1)$ in the relative interior of Δ_0.

Then $(X,\ x)$ is a simple $K3$ singularity if and only if $dim_R\Delta_0\ =\ 3$.

Thus if f is nondegenerate and defines a simple $K3$ singularity, then f_{Δ_0} is a quasi-homogeneous polynomial with a uniquely determined weights α , which called the weights of f and denoted $\alpha(f)$. We denote by $\mathbf{Q}_+$ the set of all positive rational numbers. Then $\alpha\ =\ (\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4)\ \in\ \mathbf{Q_+^4}$ and $deg_\alpha(\nu)\ :=\ \sum_{i=1}^{4}\alpha_i\nu_i\ =\ 1$ for any $\nu\ \in\ \Delta_0$. In particular, $\sum_{i=1}^{4}\alpha_i\ =\ 1$, since $(1,\ 1,\ 1,\ 1)$ is always contained in Δ_0.

We denote by $\mathbf{Z_0}$ the set of all nonnegative integer numbers.

Let $W'\ :=\ \{\alpha\ =\ (\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4)\ \in\ \mathbf{Q_+^4}\mid \alpha_1+\alpha_2+\alpha_3+\alpha_4\ =\ 1\}$ and for an element α of W', set

$$T(\alpha)\ :=\ \{\nu\ \in\ \mathbf{Z_0^4}\mid \alpha\cdot\nu\ =\ \mathbf{1}\}$$

and

$$<T(\alpha)>\ :=\ \{\sum_{\nu\in T(\alpha)} t_\nu\cdot\nu\ \in\ \mathbf{R^4}\mid \mathrm{t}_\nu\ \in\ \mathbf{R_0}\}.$$

Then the set $<T(\alpha)>$ is a closed cone in $\mathbf{R^4}$ spanned by $T(\alpha)$.

Let $W_4 := \{\alpha \in W' \mid (1, 1, 1, 1) \in \mathrm{Int} < \mathrm{T}(\alpha) >, \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4\}$. Then W_4 is the set of weights of simple $K3$ singularities. W_4 is classified, there are ninety five classes in terms of the weights of f([8]).

3 Defining equations with parameter coefficients

Yonemura listed the weights of hypersurface simple $K3$ singularities by nondegenerate polynomials and obtained the examples such that the polynomial f is quasi-homogeneous and that $\{f = 0\} \subset \mathbf{C}^4$ has a simple $K3$ singularity at the origin([8]). The minimum number of parameters in the polynomial is less than or equal to 19 and is associated with the moduli of the $K3$ surface with singularities.

Let W_4 be the set of defining equations which has a nondegenerate hypersurface simple $K3$ singularity at the origin and let $\#m(f)$ be the minimum number of parameters of the defining equation for any $f \in W_4$. Yonemura showed that there exists 3 types, 8 types, 7 types, respectively for $\#m(f) = i\ (1 \leq i \leq 3)$ in given [8].

Yonemura's results are as follows:
For $\#m(f) = 1$,

No.	The example of defining equations
f_{52}	$x^3 + y^4 + xz^3 + zw^4$
f_{56}	$x^2y + y^3z + z^5 + w^6$
f_{73}	$x^2 + y^5 + yz^5 + zw^6$

For $\#m(f) = 2$,

No.	The example of defining equations
f_{30}	$x^2 + y^5 + z^5w + w^8$
f_{46}	$x^2 + y^3 + z^{11} + zw^{12}$
f_{61}	$x^2z + y^4 + z^4w + w^7$
f_{65}	$x^2z + y^3 + z^6w + w^{11}$
f_{80}	$x^2 + y^3z + z^8w + w^{11}$
f_{84}	$x^3 + xz^3 + y^3z + yw^4 + z^2w^3$
f_{86}	$x^2y + xw^4 + y^3w + z^5 + zw^5$
f_{91}	$x^2 + y^4z + yz^5 + yw^6 + z^3w^4$

For $\#m(f) = 3$,

No.	The example of defining equations
f_{57}	$x^2y + y^4 + xz^3 + z^4w + w^6$
f_{64}	$x^2z + xy^2 + y^3w + z^6 + w^8$
f_{68}	$x^2z + y^3 + yz^5 + z^6w^2 + w^{10}$
f_{74}	$x^2 + y^4w + yz^5 + z^4w^3 + w^8$
f_{83}	$x^2 + y^3 + yw^9 + z^{10}w + z^2w^{11}$
f_{90}	$x^2 + y^4z + y^2w^5 + z^5w + zw^7$
f_{92}	$x^2 + y^3z + yw^9 + z^7w + zw^{11}$

The index number n of f_n denotes the number of the defining equation in the classification by Yonemura.

However, defining equations are not unique. So, we try to rewrite for the quasi-homogeneous polynomials.

Rewriting method We can take the following form for a weighted quasi-homogeneous polynomial f in $\mathbf{C^{n+1}}$ with the coordinate $[x_0, x_1, x_2, \ldots, x_n]$:

$$f = f_2 + \ldots + f_m$$

where f_i $(2 \leq i \leq m)$ is a homogeneous polynomial of degree i in $\mathbf{C^{n+1}}$. And let $W = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n)$ be the weights. Then we can take the following form for the homogeneous polynomial of each degree d:

$$\sum_{k_0+k_1+\ldots+k_n=d} a_{k_0 k_1 \ldots k_n} x_0^{k_0} x_1^{k_1} \ldots x_n^{k_n} \quad (k_i \in \mathbf{N_0},\ 0 \leq \mathrm{i} \leq \mathrm{n}).$$

(We denote by $\mathbf{N_0}$ the set of all positive natural numbers.)

Let $\ll$ be the lexical linear ordering of the terms of the homogeneous polynomials for $0 \leq i \leq m$ in turn from the minimal term to the maximal term given below:

Definition 5

Let $K = (k_0, k_1, \ldots, k_n)$ $(k_i \in \mathbf{N_0},\ 0 \leq \mathrm{i} \leq \mathrm{n})$ *and let* $a_K X^K$ *denote the term*

$$a_K X^K = a_{k_0 k_1 \ldots k_n} x_0^{k_0} x_1^{k_1} \ldots x_n^{k_n}.$$

Then $a_K X^K \ll b_L X^L$ *if there exists an integer* s $(0 \leq s \leq n)$ *such that* $k_i = l_i$ *for* $m = 0, 1, \ldots, s-1$ *and* $k_s < l_s$.

In the following steps, for the sake of simplicity, we shall sometimes omit the coefficients in indicating terms. We will consider the following procedure by using this ordering.

Step 1 We keep the weights $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ and try to eliminate a term X^{K_i} by a suitable analytic transformation (or linear transformation) with respect to X. We find a condition of the coefficient of term X^{K_i} where we can eliminate the term X^{K_i} without generating the term $X^{K_j} \ll X^{K_i}$. We classify the following two cases:
Case 1 : We can eliminate the term X^{K_i} without generating the term $X^{K_j} \ll X^{K_i}$.
Case 2 : Otherwise for case 1.

For the condition of case 1, we eliminate the term X^{K_i} without generating the term $X^{K_j} \ll X^{K_i}$. For the condition of case 2, we go to next step.

Step 2 We consider the next order term of the defining equation. And we do a same manipulation as step 1 for the next order term of the defining equation. By the magnification of the coordinate, we turn a coefficient of a necessary term into 1 for isolation condition in early order of this process. And moreover we simplify coefficient of all terms in this process.

For each variable x_i, one term of type x_i^n or $x_i^{n-1} x_j$ $(i \neq j)$ is necessary so that the singularity defined by the equation is isolated. We go to step 3 if we executed this manipulation for all terms.

Step 3 We simplify the coefficients of all terms finally.

Example 6
For the defining equation of $\tilde{E}_7$ singularity, we eliminate the term xyz by using analytic transformation

$$x = x' - \frac{\lambda_2 yz}{2}.$$

And we put $\lambda_2 = 2\lambda_2'$ (simplify coefficient).
Then the defining equation is as follows:

$$x'^2 + y^4 - \lambda_2'^2 y^2 z^2 + z^4.$$

For the simplicity of coefficient, we put $-\lambda_2'^2 := \lambda_2''$ (simplify coefficient, finally).
Finally, the result of rewriting is as follows:

$$x'^2 + y^4 + \lambda_2'' y^2 z^2 + z^4.$$

For the defining equation of $\tilde{E}_8$ singularity, we eliminate the term xyz by using analytic transformation

$$x = x' - \frac{\lambda_3 yz}{2}.$$

Next we eliminate the term y^2z^2 by using analytic transformation

$$y = y' + \frac{\lambda_3^2 z^2}{12}.$$

Then the defining equation is as follows:

$$x'^2 + y'^3 - \frac{\lambda_3^4}{48} y' z^4 + (1 - \frac{\lambda_3^6}{864}) z^6.$$

For the simplicity of coefficient, we put $\lambda_3 := 2\lambda_3'$ (simplify coefficient). Then the defining equation is as follows:

$$x'^2 + y'^3 - \frac{\lambda_3'^4}{3} y' z^4 + (1 - \frac{2\lambda_3'^6}{27}) z^6.$$

For the simplicity of coefficient, we put $-\lambda_3'^2 := \lambda_3''$ (simplify coefficient, finally).
Finally, the result of rewriting is as follows:

$$x'^2 + y'^3 - \frac{\lambda_3''^2}{3} y' z^4 + (1 - \frac{2\lambda_3''^3}{27}) z^6.$$

For the defining equation of $\tilde{E}_6$ singularity, we eliminate the term x^3 by using linear transformation

$$z = z' - x.$$

Next we eliminate the term $x^2y,\ xyz',\ xz'^2$ by using linear transformation

$$x = x' + \frac{\lambda_1}{6} y' + \frac{1}{2} z'',\ y = y',\ z' = z'' + \frac{\lambda_1}{3} y'.$$

Then the defining equation is as follows:

$$3x'^2 z'' + (1 + \frac{\lambda_1^3}{27}) y'^3 + \frac{\lambda_1^2}{4} y'^2 z'' + \frac{\lambda_1}{2} y' z''^2 + \frac{1}{4} z''^3.$$

For the magnification of the coordinate (we turn a coefficient of a necessary term into 1 for isolation condition), we put $x' = \frac{x''}{\sqrt{3}}$. Then the defining equation is as follows:

$$x''^2z'' + (1 + \frac{\lambda_1^3}{27})y'^3 + \frac{\lambda_1^2}{4}y'^2z'' + \frac{\lambda_1}{2}y'z''^2 + \frac{1}{4}z''^3.$$

For the magnification of the coordinate (we turn a coefficient of a necessary term into 1 for isolation condition), we put $y' = (1 + \frac{\lambda_1^3}{27})^{-\frac{1}{3}}y''$, $z'' = 2^{\frac{2}{3}}z'''$, $x'' = 2^{-\frac{1}{3}}x'''$.
Then the defining equation is as follows:

$$x'''^2z''' + y''^3 + \frac{\lambda_1^2}{2^{\frac{4}{3}}(1 + \frac{\lambda_1^3}{27})^{\frac{2}{3}}}y''^2z''' + \frac{2^{\frac{1}{3}}\lambda_1}{(1 + \frac{\lambda_1^3}{27})^{\frac{1}{3}}}y''z'''^2 + z'''^3.$$

we eliminate the term y''^2z''' by using linear transformation $y'' = y''' - \frac{\lambda_1^2}{3\ 2^{\frac{4}{3}}(1 + \frac{\lambda_1^3}{27})^{\frac{2}{3}}}z'''$.
Then the defining equation is as follows:

$$x'''^2z''' + y'''^3 + \frac{3\lambda_1(216 - \lambda_1^3)}{2^{\frac{8}{3}}(27 + \lambda_1^3)^{\frac{4}{3}}}y''z'''^2 + \frac{5832 - 540\lambda_1^3 - \lambda_1^6}{2^3(27 + \lambda_1^3)^2}z'''^3.$$

In general, for the defining equation of $\tilde{E}_6$ singularity, we can take the following:

$$x^2z = y(y - z)(y - \lambda z).$$

For this defining equation, we do similar manipulation. Then we obtain the following result.

$$x^2z + y^3 - (\frac{1}{3}(\lambda^2 - \lambda + 1))yz'^2 - (\frac{1}{27}(2\lambda^3 - 3\lambda^2 - 3\lambda + 2))z'^3.$$

(We use this equation in later example.)

By using this method, we obtain the following results.
For $\#m(f) = 1$,

No.	The defining equations
f_{52}	$x^3 + \lambda xyzw + xz^3 + y^4 + zw^4$
f_{56}	$x^2y + y^3z + \lambda yz^2w^2 + z^5 + w^6$
f_{73}	$x^2 + y^5 + \lambda y^2z^2w^2 + yz^5 + zw^6$

For $\#m(f) = 2$,

No.	The defining equations
f_{30}	$x^2 + y^5 + \lambda y^2z^2w^2 + \mu yzw^5 + z^5w + w^8$
f_{46}	$x^2 + y^3 + \lambda yz^4w^4 + z^{11} + \mu z^6w^6 + zw^{12}$
f_{61}	$x^2z + y^4 + \lambda y^2zw^2 + z^4w + \mu z^2w^4 + w^7$
f_{65}	$x^2z + y^3 + \lambda yz^2w^4 + z^6w + \mu z^3w^6 + w^{11}$
f_{80}	$x^2 + y^3z + \lambda yz^3w^4 + z^8w + \mu z^4w^6 + w^{11}$
f_{84}	$x^3 + \lambda xyzw + xz^3 + y^3z + yw^4 + \mu z^2w^3$
f_{86}	$x^2y + xw^4 + y^3w + \lambda yz^2w^2 + z^5 + \mu zw^5$
f_{91}	$x^2 + y^4z + \lambda y^2z^2w^2 + yz^5 + yw^6 + \mu z^3w^4$

For $\#m(f) = 3$,

No.	The defining equations
f_{57}	$x^2y + axz^3 + y^4 + \lambda y^2w^3 + \mu yz^2w^2 + bz^4w + w^6 (a \neq 0 \ or \ b \neq 0)$
f_{57}	$x^2y + xz^3 + y^4 + \lambda y^2w^3 + \mu yz^2w^2 + \nu z^4w + w^6 (for \ the \ above \ a \neq 0)$
f_{57}	$x^2y + y^4 + \lambda y^2w^3 + \mu yz^2w^2 + z^4w + w^6 (for \ the \ above \ a = 0)$
f_{64}	$x^2z + axy^2 + by^3w + \lambda y^2zw^2 + \mu yz^2w^3 + z^6 + \nu z^3w^4 + w^8 (a \neq 0 \ or \ b \neq 0)$
f_{64}	$x^2z + xy^2 + \lambda y^2zw^2 + \mu yz^2w^3 + z^6 + \nu z^3w^4 + w^8 (for \ the \ above \ a \neq 0)$
f_{64}	$x^2z + y^3w + \lambda yz^2w^3 + z^6 + \mu z^3w^4 + w^8 (for \ the \ above \ a = 0)$
f_{68}	$x^2z + y^3 + yz^5 + \lambda yz^2w^4 + \mu z^6w^2 + \nu z^3w^6 + w^{10}$
f_{74}	$x^2 + y^4w + \lambda y^2z^2w^2 + yz^5 + \mu yzw^5 + \nu z^4w^3 + w^8$
f_{83}	$x^2 + y^3 + \lambda yz^4w^4 + yw^9 + z^{10}w + \mu z^6w^6 + \nu z^2w^{11}$
f_{90}	$x^2 + y^4z + \lambda y^2z^2w^2 + \mu y^2w^5 + z^5w + \nu z^3w^4 + zw^7$
f_{92}	$x^2 + y^3z + \lambda yz^3w^4 + ayw^9 + z^7w + \mu z^4w^6 + bzw^{11} (a \neq 0 \ or \ b \neq 0)$
f_{92}	$x^2 + y^3z + \lambda yz^3w^4 + yw^9 + z^7w + \mu z^4w^6 + \nu zw^{11} (for \ the \ above \ a \neq 0)$
f_{92}	$x^2 + y^3z + \lambda yz^3w^4 + z^7w + \mu z^4w^6 + zw^{11} (for \ the \ above \ a = 0)$

4 Application of Gröbner Bases

In the elimination theory, one of basic strategy is Elimination Theorem. We can obtain the non-degeneracy condition of singularity at the origin from the Gröbner basis([1]). The following theorem holds.

Theorem 7 ([2])
Let $I \subset k[x_1, \ldots, x_n]$ be an ideal and let G be a Gröbner basis of I with respect to lex order where $x_1 > x_2 > \ldots > x_n$. Then, for every $0 \leq l \leq n$, the set

$$G_l = G \cap k[x_{l+1}, \ldots, x_n]$$

is a Gröbner basis of the lth elimination ideal I_l.

Let f be a defining equation, $I := < f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} >$. And let G be a Gröbner basis of I with respect to *lex* order where $x_1 > x_2 > \ldots > x_n$. Then, for every $0 \leq l \leq n$, the set

$$G_l = G \cap k[x_{l+1}, \ldots, x_n]$$

is a Gröbner basis of the lth elimination ideal I_l.

We can obtain the degeneracy condition of singularity at the origin from the Gröbner basis of the lth elimination ideal I_l. (The degeneracy condition of singularity at the origin means the singularity is non-isolated singularity at the origin.)

Example 8
For the defining equation of $\tilde{E}_7$ singularity, we calculate the non-degeneracy condition.
We put $f := x^2 + y^4 + z^4 + \lambda_2 xyz$, $I := < f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$. And let G be the Gröbner basis

of I. Then we obtain $G = \{ -(-8+\lambda_2^2)(8+\lambda_2^2)z^5, (-(-8+\lambda_2^2)(8+\lambda_2^2)yz^3, -z(\lambda_2^2 y^2 - 8z^2), z^3(8y^2 - \lambda_2^2 z^2), y(8y^2 - \lambda_2^2 z^2), 2x + \lambda_2 yz)\}$. (We use Mathematica)

From this Gröbner basis, we get that the non-degeneracy condition of this defining equation is $-64 + \lambda_2^4 \neq 0$.

5 Results

We calculate the non-degeneracy condition of moduli of simple $K3$ singularities by using the above theorem for an each defining equation. We obtain the following results. (We use Mathematica)
For $\#m(f) = 1$,

No.	The non-degeneracy conditions
f_{52}	$\lambda^4 - 256 \neq 0$
f_{56}	$\lambda^3 + 27 \neq 0$
f_{73}	$\lambda^3 + 27 \neq 0$

For $\#m(f) = 2$,

No.	The non-degeneracy conditions
f_{30}	$3125 + 16\lambda^5 + 500\lambda^2\mu - 8\lambda^4\mu^2 - 225\lambda\mu^3 + \lambda^3\mu^4 + 27\mu^5 \neq 0$
f_{46}	$(108 + 4\lambda^3 - 108\mu + 27\mu^2)(108 + 4\lambda^3 + 108\mu + 27\mu^2) \neq 0$
f_{61}	$(-2+\lambda)(2+\mu)(-8+\lambda^2-4\mu)(8+\lambda^2-4\mu) \neq 0$
f_{65}	$(108 + 4\lambda^3 - 108\mu + 27\mu^2)(108 + 4\lambda^3 + 108\mu + 27\mu^2) \neq 0$
f_{80}	$(108 + 4\lambda^3 - 108\mu + 27\mu^2)(108 + 4\lambda^3 + 108\mu + 27\mu^2) \neq 0$
f_{84}	$(16 - \lambda^2 - 18\lambda\mu + \lambda^3\mu + 27\mu^2)(-16 - \lambda^2 + 18\lambda\mu + \lambda^3\mu + 27\mu^2) \neq 0$
f_{86}	$3125 - 64\lambda^5 - 2000\lambda^2\mu - 128\lambda^4\mu^2 - 3600\lambda\mu^3 - 64\lambda^3\mu^4 - 1728\mu^5 \neq 0$
f_{91}	$\mu(27 + \lambda^3 - 36\lambda\mu - \lambda^4\mu + 8\lambda^2\mu^2 - 16\mu^3) \neq 0$

For $\#m(f) = 3$, the non-degeneracy conditions are long expressions. Therefore we write them which is not indicated by table.

f_{57} $(a \neq 0)$: $(-2+\lambda)(2+\lambda)(108 - 108\lambda + 27\lambda^2 - 16\mu^3 - 144\mu\nu + 72\lambda\mu\nu - 16\mu^2\nu^2 - 128\nu^3 + 64\lambda\nu^3)(108 + 108\lambda + 27\lambda^2 - 16\mu^3 + 144\mu\nu + 72\lambda\mu\nu - 16\mu^2\nu^2 + 128\nu^3 + 64\lambda\nu^3) \neq 0$.

f_{57} $(a = 0)$: $(-2+\lambda)(2+\lambda)(-8+4\lambda-\mu^2)(8+4\lambda-\mu^2) \neq 0$.

f_{64} $(a \neq 0)$: $(-512 + 512\lambda^2 - 128\lambda^4 + 288\lambda\mu^2 - 16\lambda^3\mu^2 + 27\mu^4 + 768\nu - 512\lambda^2\nu + 64\lambda^4\nu - 144\lambda\mu^2\nu - 384\nu^2 + 128\lambda^2\nu^2 + 64\nu^3)(512 + 512\lambda^2 + 128\lambda^4 - 288\lambda\mu^2 - 16\lambda^3\mu^2 + 27\mu^4 + 768\nu + 512\lambda^2\nu + 64\lambda^4\nu - 144\lambda\mu^2\nu + 384\nu^2 + 128\lambda^2\nu^2 + 64\nu^3) \neq 0$.

f_{64} $(a = 0)$: $(108 + 4\lambda^3 - 108\mu + 27\mu^2)(108 + 4\lambda^3 + 108\mu + 27\mu^2) \neq 0$.

f_{68} : $3125 + 16\lambda^5 + 4125\lambda^2\mu + 16\lambda^7\mu + 888\lambda^4\mu^2 + 16200\lambda\mu^3 + 16\lambda^6\mu^3 + 864\lambda^3\mu^4 + 11664\mu^5 - 5625\lambda\nu - 16\lambda^6\nu - 3420\lambda^3\mu\nu - 13500\mu^2\nu - 2592\lambda^2\mu^3\nu + 2700\lambda^2\nu^2 + 216\lambda^4\mu\nu^2 - 5670\lambda\mu^2\nu^2 + 216\lambda^3\mu^3\nu^2 - 5832\mu^4\nu^2 - 216\lambda^3\nu^3 + 6075\mu\nu^3 + 729\lambda\mu\nu^4 + 729\mu^3\nu^4 - 729\nu^5 \neq 0$.

f_{74} : $3125 + 16\lambda^5 + 500\lambda^2\mu - 8\lambda^4\mu^2 - 225\lambda\mu^3 + \lambda^3\mu^4 + 27\mu^5 - 200\lambda^3\nu - 5000\mu\nu - 16\lambda^5\mu\nu - 430\lambda^2\mu^2\nu + 8\lambda^4\mu^3\nu + 216\lambda\mu^4\nu - \lambda^3\mu^5\nu - 27\mu^6\nu + 4000\lambda\nu^2 + 16\lambda^6\nu^2 + 704\lambda^3\mu\nu^2 + 1800\mu^2\nu^2 - 8\lambda^5\mu^2\nu^2 - 296\lambda^2\mu^3\nu^2 + \lambda^4\mu^4\nu^2 + 36\lambda\mu^5\nu^2 - 192\lambda^4\nu^3 - 2560\lambda\mu\nu^3 + 64\lambda^3\mu^2\nu^3 + 32\mu^3\nu^3 - 8\lambda^2\mu^4\nu^3 + 768\lambda^2\nu^4 - 128\lambda\mu^2\nu^4 + 16\mu^4\nu^4 - 1024\nu^5 \neq 0.$

f_{83} : $3125 + 16\lambda^5 - 5625\lambda\mu - 16\lambda^6\mu + 2700\lambda^2\mu^2 - 216\lambda^3\mu^3 - 729\mu^5 + 4125\lambda^2\nu + 16\lambda^7\nu - 3420\lambda^3\mu\nu + 216\lambda^4\mu^2\nu + 6075\mu^3\nu + 729\lambda\mu^4\nu + 888\lambda^4\nu^2 - 13500\mu\nu^2 - 5670\lambda\mu^2\nu^2 + 16200\lambda\nu^3 + 16\lambda^6\nu^3 - 2592\lambda^2\mu\nu^3 + 216\lambda^3\mu^2\nu^3 + 729\mu^4\nu^3 + 864\lambda^3\nu^4 - 5832\mu^2\nu^4 + 11664\nu^5 \neq 0.$

f_{90} : $\mu(-2 + \nu)(2 + \nu)(64 - \lambda^4 - 96\lambda\mu + \lambda^5\mu + 30\lambda^2\mu^2 + \lambda^3\mu^3 + 27\mu^4 + 8\lambda^2\nu - 8\lambda^3\mu\nu + 72\mu^2\nu - \lambda^4\mu^2\nu - 36\lambda\mu^3\nu - 16\nu^2 + 16\lambda\mu\nu^2 + 8\lambda^2\mu^2\nu^2 - 16\mu^2\nu^3) \neq 0.$

f_{92} $(a \neq 0)$: $3125 + 16\lambda^5 - 5625\lambda\mu - 16\lambda^6\mu + 2700\lambda^2\mu^2 - 216\lambda^3\mu^3 - 729\mu^5 + 4125\lambda^2\nu + 16\lambda^7\nu - 3420\lambda^3\mu\nu + 216\lambda^4\mu^2\nu + 6075\mu^3\nu + 729\lambda\mu^4\nu + 888\lambda^4\nu^2 - 13500\mu\nu^2 - 5670\lambda\mu^2\nu^2 + 16200\lambda\nu^3 + 16\lambda^6\nu^3 - 2592\lambda^2\mu\nu^3 + 216\lambda^3\mu^2\nu^3 + 729\mu^4\nu^3 + 864\lambda^3\nu^4 - 5832\mu^2\nu^4 + 11664\nu^5 \neq 0.$

f_{92} $(a = 0)$: $(108 + 4\lambda^3 - 108\mu + 27\mu^2)(108 + 4\lambda^3 + 108\mu + 27\mu^2) \neq 0.$

Proof **Case:** f_{52}

$f := x^3 + \lambda xyzw + xz^3 + y^4 + zw^4$, $I := < f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w} >$. For $w = 0$, since $\frac{\partial f}{\partial y} = 0$, $y = 0$. Then, since $\frac{\partial f}{\partial z} = 0$, $x = 0$ or $z = 0$. For $x = 0$ or $z = 0$, since $\frac{\partial f}{\partial x} = 0$, $x = 0$ and $z = 0$. Therefore, for $w \neq 0$, we consider the degeneracy condition of moduli.

Let G be the Gröbner basis of I, we set the order $x > y > z > w > \lambda$. (We use Mathematica) Then

$G = \{ (-4 + \lambda)(4 + \lambda)(16 + \lambda^2) w^{10}, -(-4 + \lambda)(4 + \lambda)(16 + \lambda^2) w^9 z, (-4 + \lambda)(4 + \lambda)(16 + \lambda^2) w^5 z^3, (-4 + \lambda)(4 + \lambda)(16 + \lambda^2) w^3 z^5, (-4 + \lambda)(4 + \lambda)(16 + \lambda^2) w^2 z^6, -w(w^8 - z^7), z(-252 w^8 + \lambda^4 w^8 - 4 z^7), -w^6(4\lambda w y - 240 z^2 + \lambda^4 z^2), w^9(64 w y + \lambda^3 z^2), w^3 z(\lambda w y + 4 z^2), -w^5 z(64 w y + \lambda^3 z^2), -w^2 z^3(-192 w y + \lambda^4 w y + \lambda^3 z^2), -w^3 z^3(64 w y + \lambda^3 z^2), -w z^4(-240\lambda w y + \lambda^5 w y - 192 z^2 + \lambda^4 z^2), -w^2 z^4(64 w y + \lambda^3 z^2), -36 w^8 + 240\lambda w y z^5 - \lambda^5 w y z^5 + 228 z^7 - \lambda^4 z^7, z(-240 w^7 + \lambda^4 w^7 + 4\lambda y z^5), w^2 z(\lambda^3 w^7 + 64 y z^5), y(w^8 - z^7), -w^2(12\lambda w^2 y^2 - 144 w y z^2 + \lambda^4 w y z^2 + \lambda^3 z^4), -w^5(4 w y - \lambda z^2)(4 w y + \lambda z^2), w z(48 w^2 y^2 + \lambda^3 w y z^2 + \lambda^2 z^4), z^2(\lambda w y + z^2)(\lambda w y + 4 z^2), 144 w^7 + 228\lambda^2 w y^2 z^3 - \lambda^6 w y^2 z^3 + 180\lambda y z^5 - \lambda^5 y z^5, z(-192 w^6 + \lambda^4 w^6 - 4\lambda^2 y^2 z^3), -z(\lambda^3 w^7 - 20\lambda w y^2 z^3 - 16 y z^5), -16 w^7 y + \lambda^3 w^6 z^2 - 20\lambda y^2 z^5, -z^3(p^2 w^6 - 16 y^2 z^3), y(48 w^2 y^2 + \lambda^3 w y z^2 + \lambda^2 z^4), -z(\lambda w^5 + 4 y^3 z), -y^4 + w^4 z, 3 w^4 x - \lambda^2 w^2 y^2 z - 4\lambda w y z^3 - 3 z^5, 4 y^3 + \lambda w x z, z(12 w^3 x - \lambda^2 w y^2 z - \lambda y z^3), z(w^4 - x z^2), w^4 + \lambda w x y + 3 x z^2, (4 w^3 + \lambda x y) z, 12 w^4 y^2 + \lambda^3 w^3 y z^2 + 36 x y^2 z^2 + \lambda^2 w^2 z^4, 12 x y^3 - \lambda^2 w^2 y z^2 - \lambda w z^4, 3 x^2 + \lambda w y z + z^3 \}$

From this Gröbner basis, we obtain a necessary condition for the non-degeneracy condition of moduli in this case. It is $\lambda^4 - 256 \neq 0$. And moreover we set the other orders $x > z > y > w > \lambda, \ldots, z > y > x > w > \lambda$. Then, for their orders,

$$G_3 = G \cap \mathbf{C}[w, \lambda] = (\lambda^4 - 256) w^{10}$$

Hence, the non-degeneracy condition of moduli in this case is $\lambda^4 - 256 \neq 0$. The other cases are similar (see http://trex.h.kobe-u.ac.jp/~takahasi/sk3). ∎

6 Simplification of moduli

In the above section, we got the non-degeneracy conditions of defining equations which define hypersurface simple $K3$ singularities with unimodular, bimodular and trimodular. Same expressions exist in the non-degeneracy conditions. We need to let the coefficients of defining equations reflect states. We can realize them by regulating coefficients.

Example 9

For the defining equation of $\tilde{E}_7$ singularity, we put

$$f := x^2 + y^4 + 2\lambda_2 y^2 z^2 + z^4.$$

Then we obtain $-1 + \lambda_2^2 = 0$ as the degeneracy condition.

For the defining equation of $\tilde{E}_8$ singularity, we put

$$f := x^2 + y^3 - 3\lambda_3^2 y z^4 + 2\lambda_3^3 z^6.$$

Then we obtain $1 + 4\lambda_3^3 = 0$ as the degeneracy condition.

For the defining equation of $\tilde{E}_6$ singularity, we put

$$f := x^2 z + y^3 - \frac{\lambda_1^2 - \lambda_1 + 1}{3} y z^2 - \frac{2\lambda_1^3 - 3\lambda_1^2 - 3\lambda_1 + 2}{27} z^3.$$

The coefficients of this defining equation is not simply regulated, because, these coefficients have important meaning. $(\lambda_1 - 1)^2 \lambda_1^2 = 0$ is the degeneracy condition.

We recall moduli of elliptic curve. Then, for $x^2 z = y(y-z)(y-\lambda z)$, the j-invariant is as follows:

$$j = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2}.$$

Hence there is not j-invariant if and only if parameters of the defining equation satisfy the degeneracy condition.

For results got in the above section, we try to simplify the degenerate conditions by regulating coefficients (We regulate coefficients of defining expressions in order to simplify non-degeneracy conditions.).

Then we obtain the results as below: (We use Mathematica)

For $\#m(f) = 1$,

No.	The defining equations
f_{52}	$x^3 + 4\lambda xyzw + xz^3 + y^4 + zw^4$
f_{56}	$x^2 y + y^3 z + 3\lambda y z^2 w^2 + z^5 + w^6$
f_{73}	$x^2 + y^5 + 3\lambda y^2 z^2 w^2 + y z^5 + z w^6$

No.	The non-degeneracy conditions
f_{52}	$\lambda^4 - 1 \neq 0$
f_{56}	$\lambda^3 + 1 \neq 0$
f_{73}	$\lambda^3 + 1 \neq 0$

For $\#m(f) = 2$,

No.	The defining equations
f_{30}	$x^2 + y^5 + 5\lambda y^2z^2w^2 + 5\mu yzw^5 + z^5w + w^8$
f_{46}	$x^2 + y^3 + 3\lambda yz^4w^4 + z^{11} + 2\mu z^6w^6 + zw^{12}$
f_{61}	$x^2z + y^4 + 2\sqrt{2}\lambda y^2zw^2 + z^4w + 2\mu z^2w^4 + w^7$
f_{65}	$x^2z + y^3 + 3\lambda yz^2w^4 + z^6w + 2\mu z^3w^6 + w^{11}$
f_{80}	$x^2 + y^3z + 3\lambda yz^3w^4 + z^8w + 2\mu z^4w^6 + w^{11}$
f_{84}	$x^3 + \lambda xyzw + xz^3 + y^3z + yw^4 + \mu z^2w^3$
f_{86}	$x^2y + xw^4 + y^3w + 5\lambda yz^2w^2 + z^5 + 5\mu zw^5$
f_{91}	$x^2 + y^4z + 3\lambda y^2z^2w^2 + yz^5 + yw^6 + 3\mu z^3w^4$

No.	The non-degeneracy conditions
f_{30}	$16\lambda^5 + 20\lambda^2\mu - 40\lambda^4\mu^2 - 45\lambda\mu^3 + 25\lambda^3\mu^4 + 27\mu^5 + 1 \neq 0$
f_{46}	$(\lambda^3 + \mu^2 + 1)^2 - 4\mu^2 \neq 0$
f_{61}	$(\mu^2 - 1)((\lambda^2 - \mu)^2 - 1) \neq 0$
f_{65}	$(\lambda^3 + \mu^2 + 1)^2 - 4\mu^2 \neq 0$
f_{80}	$(\lambda^3 - \mu^2 + 1)^2 - 4\mu^2 \neq 0$
f_{84}	$(\lambda^2 - \lambda^3\mu - 27\mu^2)^2 - 4(9\lambda\mu - 8)^2 \neq 0$
f_{86}	$64\lambda^5 + 80\lambda^2\mu + 600\lambda^4\mu^2 + 720\lambda\mu^3 + 1600\lambda^3\mu^4 + 1728\mu^5 - 1 \neq 0$
f_{91}	$\mu(\lambda^3 - 12\lambda\mu - 9\lambda^4\mu + 24\lambda^2\mu^2 - 16\mu^3 + 1) \neq 0$

For $\#m(f) = 3$,

No.	The defining equations
f_{57}	$x^2y + xz^3 + y^4 + 2\lambda y^2w^3 + 3\mu yz^2w^2 + 3\nu z^4w + w^6$ $(a \neq 0)$
f_{57}	$x^2y + y^4 + 2\lambda y^2w^3 + 2\sqrt{2}\mu yz^2w^2 + z^4w + w^6$ $(a = 0)$
f_{64}	$x^2z + xy^2 + 2\lambda y^2zw^2 + 2\sqrt{2}\mu yz^2w^3 + z^6 + 2\nu z^3w^4 + w^8$ $(a \neq 0)$
f_{64}	$x^2z + y^3w + 3\lambda yz^2w^3 + z^6 + 2\mu z^3w^4 + w^8$ $(a = 0)$
f_{68}	$x^2z + y^3 + yz^5 + 5\lambda yz^2w^4 + 5\mu z^6w^2 + 5\nu z^3w^6 + w^{10}$
f_{74}	$x^2 + y^4w + 5\lambda y^2z^2w^2 + yz^5 + 5\mu yzw^5 + 5\nu z^4w^3 + w^8$
f_{83}	$x^2 + y^3 + 5\lambda yz^4w^4 + yw^9 + z^{10}w + 5\mu z^6w^6 + 5\nu z^2w^{11}$
f_{90}	$x^2 + y^4z + 2\sqrt{2}\lambda y^2z^2w^2 + 2\sqrt{2}\mu y^2w^5 + z^5w + 2\nu z^3w^4 + zw^7$
f_{92}	$x^2 + y^3z + 5\lambda yz^3w^4 + yw^9 + z^7w + 5\mu z^4w^6 + 5\nu zw^{11}$ $(a \neq 0)$
f_{92}	$x^2 + y^3z + 3\lambda yz^3w^4 + z^7w + 2\mu z^4w^6 + zw^{11}$ $(a = 0)$

For $\#m(f) = 3$, the non-degeneracy conditions are as below:

f_{57} $(a \neq 0)$: $(\lambda^2 - 1)(1 + \lambda^2 - 4\mu^3 + 12\lambda\mu\nu - 12\mu^2\nu^2 + 32\lambda\mu^3)^2 - 4(\lambda + 6\mu\nu + 16\nu^3)^2) \neq 0.$

f_{57} $(a = 0)$: $(\lambda^2 - 1)((\lambda - \mu^2)^2 - 1) \neq 0.$

f_{64} $(a \neq 0)$: $(32\lambda^2 - 16\lambda^3\mu^2 + 27\mu^4 + 24\nu + 32\lambda^4\nu - 72\lambda\mu^2\nu + 32\lambda^2\nu^2 + 8\nu^3)^2 - 64(1 + 4\lambda^4 - 9\lambda\mu^2 + 8\lambda^2\nu + 3\nu^2)^2 \neq 0.$

f_{64} $(a = 0)$: $(\lambda^3 + \mu^2 + 1)^2 - 4\mu^2 \neq 0.$

f_{68} : $16\lambda^5 + 165\lambda^2\mu + 2000\lambda^7\mu + 4440\lambda^4\mu^2 + 3240\lambda\mu^3 + 10000\lambda^6\mu^3 + 21600\lambda^3\mu^4 + 11664\mu^5 - 45\lambda\nu - 400\lambda^6\nu - 3420\lambda^3\mu\nu - 540\mu^2\nu - 12960\lambda^2\mu^3\nu + 540\lambda^2\nu^2 + 5400\lambda^4\mu\nu^2 - 5670\lambda\mu^2\nu^2 + 27000\lambda^3\mu^3\nu^2 - 29160\mu^4\nu^2 - 1080\lambda^3\nu^3 + 1215\mu\nu^3 + 3645\lambda\mu\nu^4 + 18225\mu^3\nu^4 - 729\nu^5 + 1 \neq 0.$

f_{74} : $16\lambda^5 + 20\lambda^2\mu - 40\lambda^4\mu^2 - 45\lambda\mu^3 + 25\lambda^3\mu^4 + 27\mu^5 - 40\lambda^3\nu - 40\mu\nu - 400\lambda^5\mu\nu - 430\lambda^2\mu^2\nu + 1000\lambda^4\mu^3\nu + 1080\lambda\mu^4\nu - 625\lambda^3\mu^5\nu - 675\mu^6\nu + 160\lambda\nu^2 + 2000\lambda^6\nu^2 + 3520\lambda^3\mu\nu^2 + 360\mu^2\nu^2 - 5000\lambda^5\mu^2\nu^2 - 7400\lambda^2\mu^3\nu^2 + 3125\lambda^4\mu^4\nu^2 + 4500\lambda\mu^5\nu^2 - 4800\lambda^4\nu^3 - 2560\lambda\mu\nu^3 + 8000\lambda^3\mu^2\nu^3 + 160\mu^3\nu^3 - 5000\lambda^2\mu^4\nu^3 + 3840\lambda^2\nu^4 - 3200\lambda\mu^2\nu^4 + 2000\mu^4\nu^4 - 1024\nu^5 + 1 \neq 0.$

f_{83} : $16\lambda^5 - 45\lambda\mu - 400\lambda^6\mu + 540\lambda^2\mu^2 - 1080\lambda^3\mu^3 - 729\mu^5 + 165\lambda^2\nu + 2000\lambda^7\nu - 3420\lambda^3\mu\nu + 5400\lambda^4\mu^2\nu + 1215\mu^3\nu + 3645\lambda\mu^4\nu + 4440\lambda^4\nu^2 - 540\mu\nu^2 - 5670\lambda\mu^2\nu^2 + 3240\lambda\nu^3 + 10000\lambda^6\nu^3 - 12960\lambda^2\mu\nu^3 + 27000\lambda^3\mu^2\nu^3 + 18225\mu^4\nu^3 + 21600\lambda^3\nu^4 - 29160\mu^2\nu^4 + 11664\nu^5 + 1 \neq 0.$

f_{90} : $\mu(\nu^2 - 1)(\lambda^4 + 12\lambda\mu - 8\lambda^5\mu - 30\lambda^2\mu^2 - 8\lambda^3\mu^3 - 27\mu^4 - 2\lambda^2\nu + 16\lambda^3\mu\nu - 18\mu^2\nu + 16\lambda^4\mu^2\nu + 72\lambda\mu^3\nu + \nu^2 - 8\lambda\mu\nu^2 - 32\lambda^2\mu^2\nu^2 + 16\mu^2\nu^3 - 1) \neq 0.$

f_{92} $(a \neq 0)$: $16\lambda^5 - 45\lambda\mu - 400\lambda^6\mu + 540\lambda^2\mu^2 - 1080\lambda^3\mu^3 - 729\mu^5 + 165\lambda^2\nu + 2000\lambda^7\nu - 3420\lambda^3\mu\nu + 5400\lambda^4\mu^2\nu + 1215\mu^3\nu + 3645\lambda\mu^4\nu + 4440\lambda^4\nu^2 - 540\mu\nu^2 - 5670\lambda\mu^2\nu^2 + 3240\lambda\nu^3 + 10000\lambda^6\nu^3 - 12960\lambda^2\mu\nu^3 + 27000\lambda^3\mu^2\nu^3 + 18225\mu^4\nu^3 + 21600\lambda^3\nu^4 - 29160\mu^2\nu^4 + 11664\nu^5 + 1 \neq 0.$

f_{92} $(a = 0)$: $(\lambda^3 + \mu^2 + 1)^2 - 4\mu^2 \neq 0.$

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