Communications of JSSAC (2012)
Vol. 1, pp. 67 – 74

The Bell locus of rational functions and problems of Goldberg

Masayo Fujimura*   Mohaby Karima†   Masahiko Taniguchi‡§

National Defense Academy   Nara Women's University   Nara Women's University

(Received 04/Feb/2010   Accepted 26/Apr/2010)

Abstract

The spaces of Möbius equivalence classes of rational functions can be represented by the Bell loci. Noting this fact, we solve problems of Goldberg, namely, determine several kinds of the non-generic loci for the map from the Bell locus CB_d to the space of the sets of critical points explicitly when the degree is small. The symbolic and algebraic computation system is crucial for the results.

1 Introduction

A general form of a rational function of degree d is

$$\frac{P(z)}{Q(z)}$$

with polynomials $P(z)$ and $Q(z)$ of degree at most d, where $P(z)$ and $Q(z)$ have no common non-constant factors and one of them has d as the degree, and the *canonical family* C_d of rational functions of degree d is defined as the totality of those with monic $Q(z)$ of degree d:

$$\left\{R(z) = \frac{P(z)}{Q(z)} \ : \ \deg Q = d, \ \mathrm{Resul}(P, Q) \neq 0, \ Q \text{ is monic}\right\}.$$

Writing

$$P(z) = p_d z^d + \cdots + p_0, \qquad Q(z) = z^d + q_{d-1}z^{d-1} + \cdots + q_0,$$

we call the vector $(p_d, \cdots, p_0, q_{d-1}, \cdots, q_0)$ the system of *coefficient parameters* for C_d. Also see Theorem 2.4 in [5].

On the other hand, from the viewpoint of the geometric function theory, special attentions have been paid to rational functions of the overlap locus whose overlap type contains an integer not

*masayo@nda.ac.jp

†iam_ karima@cc.nara-wu.ac.jp

‡The third author is partially supported by Grant-in-Aid for Scientific Research (C) 19540181.

§tanig@cc.nara-wu.ac.jp

less than 3. (Cf. Definition 3.7 and Theorem 3.8 in [5].) Actually, every non-degenerate d-ply connected planar domain can be mapped biholomorphically onto a domain

$$W_{\mathbf{w}} := \left\{ z \in \mathbb{C} \ : \left| \ z + \sum_{k=1}^{d-1} \frac{u_k}{z - v_k} \right| < 1 \right\}$$

with some complex vectors $\mathbf{w} = (u_1, \ldots, u_{d-1}, v_1, \ldots, v_{d-1})$. See, [7]. Such domains $W_{\mathbf{w}}$ are a *new* type of canonical planar domains with connectivity d, which are called *Bell representations*.

Here recall that, every overlap locus $C\{n_1, \cdots, n_p\}$ admits so-called decomposition parameters: Letting $\hat{P}(z) = P(z) - zQ(z)$ and assuming that ∞ is not a fixed point, $Q(z)/\hat{P}(z)$ has a unique partial fractions decomposition

$$\frac{\alpha_{1,n_1}}{(z-\zeta_1)^{n_1}} + \cdots + \frac{\alpha_{1,1}}{z-\zeta_1} + \frac{\alpha_{2,n_2}}{(z-\zeta_2)^{n_2}} + \cdots + \frac{\alpha_{p,1}}{z-\zeta_p},$$

where, ζ_k are fixed points of $R(z)$, $\alpha_{k,n_k} \neq 0$ for every k, and $\sum_{k=1}^{p} \alpha_{k,1} = 1$. The set $\{\zeta_k\}$ of fixed points and the set $\{\alpha_{k,\ell}\}$ of coefficients give a system of coordinates for $C\{n_1, \cdots, n_p\}$, and is called the system of *decomposition parameters* for $C\{n_1, \cdots, n_p\}$.

The *Bell locus* CB_d is the union of the overlap locus $C\{n_1, \cdots, n_p\}$ with $n_1 \geq 3$ such that the corresponding $\zeta_1 = \infty$. Bell representations are those in the Bell locus CB_d with generic polar divisors.

Proposition 1
The Bell locus CB_d has a system of coordinates consisting of coefficients in the representation

$$z + \frac{\hat{P}(z)}{Q(z)},$$

with

$$\hat{P}(z) = a_{d-2} z^{d-2} + \cdots + a_0, \quad Q(z) = z^{d-1} + b_{d-2} z^{d-2} + \cdots + b_0.$$

Next, we state Goldberg's theorem.

Definition 2
We say that two rational functions R_1 and R_2 are *Möbius equivalent* if there is a Möbius transformation $M : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

$$R_2 = M \circ R_1.$$

We denote by X_d the set of all Möbius equivalence classes of rational functions of degree d.

Then Goldberg showed

Proposition 3 ([6] Thoerem 1.3)
Every $(2d-2)$-tuple B of points in $\widehat{\mathbb{C}}$ is the set of critical points of at most $C(d)$ classes in X_d, where $C(d)$ is the Catalan number:

$$C(d) = \frac{1}{d} \begin{pmatrix} 2d-2 \\ d-1 \end{pmatrix}.$$

The maximal is attained on a Zariski open subset of the space $\widehat{\mathbb{C}}^{2d-2}$ of all B.

Remark 4
Every set of critical points of a rational function is *admissible*, namely, every point has multiplicity at most $d-1$.

In the sequel, we will assume that ∞ is non-critical, for the other cases can be treated similarly and more easily. Then, the Bell locus represents the corresponding subset of X_d faithfully.

Proposition 5
Let X'_d is the sublocus of X_d consisting of those classes of rational functions such that ∞ is non-critical. Then every point in X'_d is represented by a point in CB_d.
Moreover, different functions in CB_d correspond to different points in X'_d.

Consider a polynomial map Φ_d of CB_d to $\mathbb{C}^{2d-2}$ defined from the equation

$$Q^2(z) + \hat{P}'(z)Q(z) - \hat{P}(z)Q'(z) = z^{2d-2} + c_{2d-3}z^{2d-3} + \cdots + c_1 z + c_0$$

by sending

$$(\mathbf{a}, \mathbf{b}) = (a_{d-2}, \cdots, a_0, b_{d-2}, \cdots, b_0)$$

to

$$(\mathbf{c}) = (c_{2d-3}, \cdots, c_0).$$

Then Goldberg's theorem implies that Φ_d is $C(d)$-valent on a Zariski open subset of $\mathbb{C}^{2d-2}$. And the problems in [6] 143p mean

Problem 6
1) Describe in detail the ramification set of the map Φ_d.
2) For every point $(\mathbf{c})$ in $\mathbb{C}^{2d-2}$, determine the number of points in the preimage $\Phi_d^{-1}((\mathbf{c}))$.

Note that to solve these problems are actually very important. For instance, Eremenko and Gabrielov proves the Shapiro conjecture for two-dimensional case by using the fact that, for any given $2d-2$ distinct points on the real line, there exist *exactly* $C(d)$ distinct Möbius equivalent classes of rational functions of degree d with these critical points. See [3] and [4]. (Also cf. [8].)

In the next section, we will determine for Φ_d, the ramification locus, the exceptional locus, and the degeneration locus where the number of points in the preimage by Φ_d is less than $C(d)$, for $d = 2, 3$, and 4 explicitly. Here the exceptional locus $E(d)$ is the set of all non-admissible points $(\mathbf{c}) \in \mathbb{C}^{2d-2}$. Recall that Goldberg also asked whether there is a rational function having an arbitrarily given admissible set of points as the set of critical points. We answer this question for $d = 2, 3$, and 4.

2 The explicit descriptions of the loci for the map Φ_d

In this section, we use "risa/asir", a symbolic and algebraic computation system, to obtain the defining equations of the loci considered.

Definition 7
Let $\hat{P}(z)$ and $Q(z)$ be as before. Set

$$R(d) = \left\{(\mathbf{a}, \mathbf{b}) \in \mathbb{C}^{2d-2} \ : \ \mathrm{Resul}(\hat{P}, Q) = 0\right\},$$

which is the locus where Φ_d is not defined. (In other words, CB_d can be identified with $\mathbb{C}^{2d-2} - R(d)$.)

First, we begin with

Proposition 8
If $d = 2$, then the map $\Phi_2 : CB_2 \to \mathbb{C}^2 - E(2)$ is bijective, and the exceptional locus $E(2)$ is the algebraic curve defined by $c_1^2 - 4c_0 = 0$.

Proof The map Φ_2 is given by $(a_0, b_0) \mapsto (b_0^2 - a_0, 2b_0)$. The locus $R(2)$ is given by $a_0 = 0$ and corresponds to the locus $E(2)$ defined by $c_1^2 - 4c_0 = 0$. Conversely, each point $(\mathbf{c})$ on the locus $c_1^2 - 4c_0 = 0$ coincides with a non-admissible set, because the equation $z^2 + c_1 z + \frac{c_1^2}{4} = 0$ always has a double root. Therefore $c_1^2 - 4c_0 = 0$ gives a defining equation of $E(2)$. ∎

If $d = 3$, we have the following.

Theorem 9
If $d = 3$, the ramification locus of Φ_3 is

$$a_1 = b_1^2 - 4b_0,$$

$\Phi_3(CB_3) = \mathbb{C}^4 - E(3)$, and Φ_3 is 2-valent on the set of points in $\mathbb{C}^4 - E(3)$ satisfying that

$$c_2^2 - 3c_1c_3 + 12c_0 \neq 0, \quad E_0 \neq 0.$$

Moreover, the exceptional locus $E(3)$ is the algebraic variety defined by $E_0 = E_1 = 0$. Here

$$\begin{aligned}
E_1 &= 216c_1^2 - 72c_2c_3c_1 + (216c_3^2 - 576c_2)c_0 + 16c_2^3, \\
E_0 &= -27c_1^4 + (-4c_3^3 + 18c_2c_3)c_1^3 + ((-6c_3^2 + 144c_2)c_0 + c_2^2c_3^2 - 4c_2^3)c_1^2 \\
&\quad +(-192c_3c_0^2 + (18c_2c_3^3 - 80c_2^2c_3)c_0)c_1 + 256c_0^3 \\
&\quad +(-27c_3^4 + 144c_2c_3^2 - 128c_2^2)c_0^2 + (-4c_2^3c_3^2 + 16c_2^4)c_0.
\end{aligned}$$

Proof The Jacobian of the map Φ_3 is

$$\begin{vmatrix} 0 & 0 & 2 & 0 \\ -1 & 0 & 2b_1 & 2 \\ 0 & -2 & 2b_0 & 2b_1 \\ b_0 & -b_1 & -a_0 & a_1 + 2b_0 \end{vmatrix} = 4(a_1 - b_1^2 + 4b_0).$$

Also, for $(\mathbf{c}) = (c_3, c_2, c_1, c_0)$ in $\mathbb{C}^4 - E(3)$, every (a_1, a_0, b_1, b_0) in $\Phi_3^{-1}((\mathbf{c}))$ is a solution of

$$\begin{cases} 12b_0^2 - 4c_2b_0 + c_1c_3 - 4c_0 = 0 \\ b_1 = \frac{1}{2}c_3 \\ a_1 = \frac{1}{4}(8b_0 + c_3^2 - 4c_2) \\ a_0 = \frac{1}{2}(c_3b_0 - c_1) \end{cases} \tag{1}$$

which has exactly 2 solutions except for the set defined by the discriminant

$$c_2^2 - 3c_1c_3 + 12c_0 = 0.$$

Here, the locus $R(3)$ is given by

$$a_0^2 - b_1a_1a_0 + a_1^2b_0 = 0. \tag{2}$$

Eliminating four variables a_0, a_1, b_0, b_1 from the equation $r = a_0^2 - b_1a_1a_0 + a_1^2b_0$ by using (1), we have the following equation

$$-432r^2 + E_1r + E_0 = 0, \tag{3}$$

where E_0, E_1 are as in the theorem. Then, $R(3)$ corresponds to the condition that (3) has 0 as a solution, and there are no rational functions of degree 3 corresponding to $(\mathbf{c})$ if and only if the equation (3) has 0 as a unique solution. By using the relation between coefficients and solutions, we can check that the equation $z^4 + c_3z^3 + c_2z^2 + c_1z + c_0 = 0$ has a solution of multiplicity at least 3, for every point $(\mathbf{c})$ in the set defined by $E_0 = E_1 = 0$. Therefore $E_0 = E_1 = 0$ gives a set of defining equations of $E(3)$. ∎

Theorem 10
If $d = 4$, the ramification locus of Φ_4 is given by

$$(b_2a_2 - b_2^3 + 4b_1b_2 - 9b_0)a_1 - b_1a_2^2 + (-3a_0 + b_1b_2^2 + 6b_0b_2 - 5b_1^2)a_2 \\ +(b_2^2 - 3b_1)a_0 - 4b_0b_2^3 + b_1^2b_2^2 + 18b_0b_1b_2 - 4b_1^3 - 27b_0^2 = 0.$$

Moreover, for a given $(\mathbf{c})$ in $\mathbb{C}^6 - E(4)$, b_1 is a solution of algebraic equation of degree 5, and other coefficients are determined from b_1 and $(\mathbf{c})$ uniquely.

In particular, $\Phi_4(CB_4) = \mathbb{C}^6 - E(4)$, and Φ_4 is 5-valent on the set of points in $\mathbb{C}^6 - E(4)$ satisfying $D \neq 0$ and $E_0 \neq 0$, where D is the discriminant given explicitly in the proof and E_0 is stated in Remark 12, where the description of the exceptional set $E(4)$ is given.

Proof The Jacobian of the map Φ_4 is

$$8((b_2a_2 - b_2^3 + 4b_1b_2 - 9b_0)a_1 - b_1a_2^2 + (-3a_0 + b_1b_2^2 + 6b_0b_2 - 5b_1^2)a_2 \\ +(b_2^2 - 3b_1)a_0 - 4b_0b_2^3 + b_1^2b_2^2 + 18b_0b_1b_2 - 4b_1^3 - 27b_0^2).$$

For $(\mathbf{c})$ in $\mathbb{C}^6 - E(4)$, every $(a_2, a_1, a_0, b_2, b_1, b_0)$ in $\Phi_4^{-1}((\mathbf{c}))$ is a solution of

$$\left\{\begin{array}{rl} B_1 = & 1296b_1^5 - 1296c_4b_1^4 + (216c_5c_3 - 432c_2 + 432c_4^2)b_1^3 \\ & +(-144c_4c_5c_3 + (24c_5^2 + 288c_4)c_2 - 216c_5c_1 + 1296c_0 - 48c_4^3)b_1^2 \\ & +(9c_5^2c_3^2 + (-36c_5c_2 + 108c_1 + 24c_4^2c_5)c_3 + (-8c_4c_5^2 - 48c_4^2)c_2 \\ & +(-4c_5^3 + 72c_4c_5)c_1 + (144c_5^2 - 864c_4)c_0)b_1 - 3c_4c_5^2c_3^2 \\ & +((2c_5^3 + 12c_4c_5)c_2 + (-6c_5^2 - 36c_4)c_1)c_3 - 8c_5^2c_2^2 + 48c_5c_1c_2 \\ & -72c_1^2 + (4c_5^4 - 48c_4c_5^2 + 144c_4^2)c_0 = 0 \\ B_{0,1} = & -12c_5b_1^2 + (72b_0 + 4c_4c_5)b_1 + (4c_5^2 - 24c_4)b_0 - 12c_1 + 4c_5c_2 - c_5^2c_3 = 0 \\ B_{0,2} = & -12b_1^3 + 4c_4b_1^2 + (4c_5b_0 + 4c_2 - c_5c_3)b_1 + 24b_0^2 - 6c_3b_0 - 12c_0 = 0 \\ b_2 = & \frac{1}{2}c_5 \\ a_2 = & \frac{1}{4}(8b_1 + c_5^2 - 4c_4) \\ a_1 = & \frac{1}{2}(c_5b_1 + 2b_0 - c_3) \\ a_0 = & \frac{1}{12}(12b_1^2 - 4c_4b_1 + 2c_5b_0 + c_5c_3 - 4c_2). \end{array}\right. \tag{4}$$

There is a unique common root b_0 of $B_{0,1} = B_{0,2} = 0$ if and only if b_1 is a solution of $B_1 = 0$, since

$$\mathrm{Resul}_{b_0}(B_{0,1},\ B_{0,2}) = -48B_1.$$

Therefore, there are five solutions of $B_1 = 0$ except for the locus defined by the following discriminant D:

$(6c_5^3c_4 - c_5^5 - 27c_3c_5^2 + 108c_2c_5 - 324c_1)^2 \times 12(108c_0c_1^2c_4 - 648c_0^2c_1c_5 + (-108c_0c_1c_2 - 27c_1^3)c_3 + 32c_0c_2^3 + 9c_1^2c_2^2)c_4^6 + 4(2916c_0^3c_5^2 + ((972c_0^2c_2 - 1863c_0c_1^2)c_3 - 234c_0c_1c_2^2 + 27c_1^3c_2)c_5 + 81c_0c_1c_3^3 + (-27c_0c_2^2 + 81c_1^2c_2)c_3^2 + (-51c_1c_2^3 + 2916c_0^2c_1)c_3 + 8c_2^5 - 2592c_0^2c_2^2 + 5022c_0c_1^2c_2 - 162c_1^4)c_4^5 + (49572c_0^2c_1c_3 +$

$$
\begin{array}{l}
108c_0^2c_2^2 + 4284c_0c_1^2c_2 + 27c_1^4)c_5^2 + (-972c_0^2c_3^3 + (8316c_0c_1c_2 + 2106c_1^3)c_3^2 + (-2412c_0c_2^3 - 738c_1^2c_2^2 - \\
34992c_0^3)c_3 + 8c_1c_2^4 - 92016c_0^2c_1c_2 - 42444c_0c_1^3)c_5 + -81c_1^2c_3^4 + 54c_1c_2^2c_3^3 + (-9c_2^4 + 1944c_0^2c_2 - \\
4860c_0c_1^2)c_3^2(-13500c_0c_1c_2^2 - 3888c_1^3c_2)c_3 + 4320c_0c_2^4 + 1320c_1^2c_2^3 + 93312c_0^3c_2 - 329508c_0^2c_1^2)c_4^4 - \\
4((18225c_0^3c_3 + 6615c_0^2c_1c_2 - 849c_0c_1^3)c_5^3 + ((6156c_0^2c_2 - 2322c_0c_1^2)c_3^2 + (-378c_0c_1c_2^2 + 450c_1^3c_2)c_3 - \\
330c_0c_2^4 - 92c_1^2c_2^3 - 31590c_0^3c_2 - 117693c_0^2c_1^2)c_5^2 + (486c_0c_1c_3^4 + (-162c_0c_2^2 + 513c_1^2c_2)c_3^3 + (-324c_1c_2^3 + \\
15795c_0^2c_1)c_3^2 + (51c_2^5 - 27621c_0^2c_2^2 + 7182c_0c_1^2c_2 - 3645c_1^4)c_3 + 6030c_0c_1c_2^3 + 531c_1^3c_2^2 - 435942c_0^3c_1)c_5 + \\
(-972c_0c_1c_2 - 81c_1^3)c_3^3 + (324c_0c_2^3 - 918c_1^2c_2^2 + 2187c_0^3)c_3^2 + (603c_1c_2^4 - 99387c_0^2c_1c_2 - 57105c_0c_1^3)c_3 - \\
96c_2^6 + 50544c_0^2c_2^3 - 6696c_0c_1^2c_2^2 + 2025c_1^4c_2 + 69984c_0^4)c_4^3 + 2((20250c_0^3c_2 - 24975c_0^2c_1^2)c_5^4 + \\
(-43335c_0^2c_1c_3^2 + (6345c_0^2c_2^2 - 6642c_0c_1^2c_2 - 492c_1^4)c_3 - 1062c_0c_1c_2^3 + 317c_1^3c_2^2 - 814050c_0^3c_1)c_5^3 + \\
(2916c_0^2c_3^4 + (-6804c_0c_1c_2 - 1917c_1^3)c_3^3 + (1836c_0c_2^3 + 1296c_1^2c_2^2 + 102060c_0^3)c_3^2 + (-369c_1c_2^4 + \\
6804c_0^2c_1c_2 - 19224c_0c_1^3)c_3 + 54c_2^6 + 13392c_0^2c_2^3 + 57672c_0c_1^2c_2^2 - 2610c_1^4c_2 - 1202850c_0^4)c_5^2 + (243c_1^2c_3^5 - \\
162c_1c_2^2c_3^4 + (27c_2^4 - 11664c_0^2c_2 + 8262c_0c_1^2)c_3^3 + (22680c_0c_1c_2^2 + 1512c_1^3c_2)c_3^2 + (-6750c_0c_2^4 + 756c_1^2c_2^3 - \\
858762c_0^3c_2 - 536301c_0^2c_1^2)c_3 - 468c_1c_2^5 + 280260c_0^2c_1c_2^2 - 246780c_0c_1^3c_2 + 10125c_1^5)c_5 - 486c_1^2c_2c_3^4 + \\
(324c_1c_2^3 - 32805c_0^2c_1)c_3^3 + (-54c_2^5 + 22599c_0^2c_2^2 - 134622c_0c_1^2c_2 - 18225c_1^4)c_3^2 + (55242c_0c_1c_2^3 + \\
6345c_1^3c_2^2 - 1167858c_0^3c_1)c_3 - 5184c_0c_2^5 + 54c_1^2c_2^4 + 1562976c_0^3c_2^2 - 733860c_0^2c_1^2c_2 + 344250c_0c_1^4)c_4^2 + \\
4(50625c_0^3c_1c_5^5 + (30375c_0^3c_3^2 + (29700c_0^2c_1c_2 + 3060c_0c_1^3)c_3 - 2025c_0^2c_2^3 - 1305c_0c_1^2c_2^2 + 88c_1^4c_2 + \\
455625c_0^4)c_5^4 + ((9720c_0^2c_2 + 1539c_0c_1^2)c_3^3 + (756c_0c_1c_2^2 + 1386c_1^3c_2)c_3^2 + (-972c_0c_2^4 - 450c_1^2c_2^3 + \\
24300c_0^3c_2 - 61155c_0^2c_1^2)c_3 + 27c_1c_2^5 - 123390c_0^2c_1c_2^2 + 4698c_0c_1^3c_2 - 600c_1^5)c_5^3 + (729c_0c_1c_3^5 + \\
(-243c_0c_2^2 + 810c_1^2c_2)c_3^4 + (-513c_1c_2^3 + 38637c_0^2c_1)c_3^3 + (81c_2^5 - 67311c_0^2c_2^2 + 5670c_0c_1^2c_2 - 4590c_1^4)c_3^2 + \\
(-7182c_0c_1c_2^3 - 3321c_1^3c_2^2 + 798255c_0^3c_1)c_3 + 5022c_0c_2^5 + 1071c_1^2c_2^4 - 366930c_0^3c_2^2 + 748278c_0^2c_1^2c_2 - \\
12825c_0c_1^4)c_5^2 + ((-2916c_0c_1c_2 + 243c_1^3)c_3^4 + (972c_0c_2^3 - 3402c_1^2c_2^2 + 59049c_0^3)c_3^3 + (2079c_1c_2^4 - \\
1458c_0^2c_1c_2 + 20655c_0c_1^3)c_3^2 + (-324c_2^6 + 99387c_0^2c_2^3 + 3402c_0c_1^2c_2^2 + 29700c_1^4c_2 + 2066715c_0^4)c_3 - \\
23004c_0c_1c_2^4 - 6615c_1^3c_2^3 - 538002c_0^3c_1c_2 - 1148175c_0^2c_1^3)c_5 + 13122c_0c_1^2c_3^4 + (-5832c_0c_1c_2^2 + \\
9720c_1^3c_2)c_3^3 + (486c_0c_2^4 - 6156c_1^2c_2^3 - 118098c_0^3c_2 + 557685c_0^2c_1^2)c_3^2 + (972c_1c_2^5 - 429381c_0^2c_1c_2^2 + \\
24300c_0c_1^3c_2 - 50625c_1^5)c_3 + 23328c_0^2c_2^4 + 31590c_0c_1^2c_2^3 + 10125c_1^4c_2^2 - 5196312c_0^4c_2 + 3936600c_0^3c_1^2)c_4 - \\
253125c_0^4c_5^6 - 2((101250c_0^3c_2 + 13500c_0^2c_1^2)c_3 - 10125c_0^2c_1c_2^2 + 1200c_0c_1^3c_2 - 32c_1^5)c_5^5 + 3(4050c_0^2c_1c_3^3 + \\
(-12150c_0^2c_2^2 - 6120c_0c_1^2c_2 - 144c_1^4)c_3^2 + (4860c_0c_1c_2^3 - 328c_1^3c_2^2 + 243000c_0^3c_1)c_3 - 216c_0c_2^5 + 9c_1^2c_2^4 + \\
229500c_0^3c_2^2 - 17100c_0^2c_1^2c_2 + 3360c_0c_1^4)c_5^4 - 6(1458c_0^2c_3^5 + (-162c_0c_1c_2 - 162c_1^3)c_3^4 + (-54c_0c_2^3 + \\
639c_1^2c_2^2 + 72900c_0^3)c_3^3 + (-351c_1c_2^4 - 13770c_0^2c_1c_2 - 10908c_0c_1^3)c_3^2 + (54c_2^6 - 38070c_0^2c_2^3 + 6408c_0c_1^2c_2^2 - \\
2040c_1^4c_2 + 820125c_0^4)c_3 + 7074c_0c_1c_2^4 - 566c_1^3c_2^3 + 765450c_0^3c_1c_2 - 1215c_0^2c_1^3)c_5^3 - 27(27c_1^2c_3^6 - \\
18c_1c_2^2c_3^5 + (3c_2^4 - 1944c_0^2c_2 + 1512c_0c_1^2)c_3^4 + (-612c_0c_1c_2^2 - 228c_1^3c_2)c_3^3 + (180c_0c_2^4 - 344c_1^2c_2^3 - \\
82620c_0^3c_2 + 15876c_0^2c_1^2)c_3^2 + (276c_1c_2^5 + 39726c_0^2c_1c_2^2 + 9060c_0c_1^3c_2 + 1000c_1^5)c_3 - 48c_2^7 + 12204c_0^2c_2^4 - \\
17436c_0c_1^2c_2^3 + 1850c_1^4c_2^2 - 583200c_0^4c_2 - 281880c_0^3c_1^2)c_5^2 + 162(18c_1^2c_2c_3^5 + (-12c_1c_2^3 - 729c_0^2c_1)c_3^4 + \\
(2c_2^5 - 405c_0^2c_2^2 + 954c_0c_1^2c_2 + 75c_1^4)c_3^3 + (-390c_0c_1c_2^3 - 535c_1^3c_2^2 - 32076c_0^3c_1)c_3^2 + (72c_0c_2^5 + 306c_1^2c_2^4 - \\
14418c_0^3c_2^2 + 19710c_0^2c_1^2c_2 + 4500c_0c_1^4)c_3 - 48c_1c_2^6 + 10764c_0^2c_1c_2^3 - 10050c_0c_1^3c_2^2 + 1250c_1^5c_2 - \\
262440c_0^4c_1)c_5 - 81(108c_1^3c_3^5 - 72c_1^2c_2^2c_3^4 + (12c_1c_2^4 - 2916c_0^2c_1c_2 + 5400c_0c_1^3)c_3^3 + (108c_0^2c_2^3 - \\
2520c_0c_1^2c_2^2 - 1500c_1^4c_2 - 19683c_0^4)c_3^2 + (432c_0c_1c_2^4 + 900c_1^3c_2^3 - 102060c_0^3c_1c_2 + 60750c_0^2c_1^3)c_3 - \\
144c_1^2c_2^5 + 3456c_0^3c_2^3 + 29700c_0^2c_1^2c_2^2 - 22500c_0c_1^4c_2 + 3125c_1^6 - 629856c_0^5) = 0. \qquad \blacksquare
\end{array}
$$

Remark 11

If $d = 5$, then we can see that b_2 is a solution of an algebraic equation of degree 14 for every $(\mathbf{c})$ in $\mathbb{C}^8 - E(5)$.

Remark 12

The locus $R(4)$ is given by $r = 0$, where

$$
\begin{array}{l}
r = b_0^2a_2^3 + (-b_0b_1a_1 + (-2b_0b_2 + b_1^2)a_0)a_2^2 + (b_0b_2a_1^2 + (-b_1b_2 + 3b_0)a_0a_1 \\
\quad +(b_2^2 - 2b_1)a_0^2)a_2 - b_0a_1^3 + b_1a_0a_1^2 - b_2a_0^2a_1 + a_0^3.
\end{array} \tag{5}
$$

Eliminating six variables in $(\mathbf{a}, \mathbf{b})$ from the equation (5) by using (4), we have the following equation

$$\begin{aligned} &34828517376r^5 + 5038848E_4r^4 + 186624E_3r^3 - 864E_2r^2 \\ &+16E_1E_0r - E_0^2 = 0, \end{aligned} \tag{6}$$

where each E_j $(j = 0, 1, 2, 3, 4)$ is a polynomial in $\mathbb{C}[c_0, \cdots, c_5]$. And, there are no rational functions of degree 4 corresponding to $(\mathbf{c})$ if and only if the equation (6) has 0 as a unique solution.

On the other hand, by using the relation between coefficients and solutions, we can check that the equation $z^6 + c_5z^5 + c_4z^4 + c_3z^3 + c_2z^2 + c_1z + c_0 = 0$ has a solution of multiplicity at least 4, for every point $(\mathbf{c})$ in the set defined by $E_4 = E_3 = E_2 = E_0 = 0$. Therefore $E_4 = E_3 = E_2 = E_0 = 0$ gives a set of defining equations of $E(4)$.

Finally, E_0 is given by

$$\begin{aligned}
&3125c_0^4c_5^6 + (-2500c_0^3c_1c_4 + (-3750c_0^3c_2 + 2000c_0^2c_1^2)c_3 + 2250c_0^2c_1c_2^2 - 1600c_0c_1^3c_2 + 256c_1^5)c_5^5 + \\
&((2000c_0^3c_2 - 50c_0^2c_1^2)c_4^2 + (2250c_0^3c_3^2 + (-2050c_0^2c_1c_2 + 160c_0c_1^3)c_3 - 900c_0^2c_2^3 + 1020c_0c_1^2c_2^2 - 192c_1^4c_2 - \\
&22500c_0^4)c_4 - 900c_0^2c_1c_3^3 + (825c_0^2c_2^2 + 560c_0c_1^2c_2 - 128c_1^4)c_3^2 + (-630c_0c_1c_2^3 + 144c_1^3c_2^2 + 2250c_0^3c_1)c_3 + \\
&108c_0c_2^5 - 27c_1^2c_2^4 + 1500c_0^3c_2^2 - 1700c_0^2c_1^2c_2 + 320c_0c_1^4)c_5^4 + ((-1600c_0^3c_3 + 160c_0^2c_1c_2 - 36c_0c_1^3)c_4^3 + \\
&(1020c_0^2c_1c_3^2 + (560c_0^2c_2^2 - 746c_0c_1^2c_2 + 144c_1^4)c_3 + 24c_0c_1c_2^3 - 6c_1^3c_2^2 + 15600c_0^3c_1)c_4^2 + ((-630c_0^2c_2 + \\
&24c_0c_1^2)c_3^3 + (356c_0c_1c_2^2 - 80c_1^3c_2)c_3^2 + (-72c_0c_2^4 + 18c_1^2c_2^3 + 19800c_0^3c_2 - 12330c_0^2c_1^2)c_3 - 13040c_0^2c_1c_2^2 + \\
&9768c_0c_1^3c_2 - 1600c_1^5)c_4 + 108c_0^2c_3^5 + (-72c_0c_1c_2 + 16c_1^3)c_3^4 + (16c_0c_2^3 - 4c_1^2c_2^2 - 1350c_0^3)c_3^3 + (1980c_0^2c_1c_2 - \\
&208c_0c_1^3)c_3^2 + (-120c_0^2c_2^3 - 682c_0c_1^2c_2^2 + 160c_1^4c_2 + 27000c_0^4)c_3 + 144c_0c_1c_2^4 - 36c_1^3c_2^3 - 1800c_0^3c_1c_2 + \\
&410c_0^2c_1^3)c_5^3 + (256c_0^3c_4^5 + (-192c_0^2c_1c_3 - 128c_0^2c_2^2 + 144c_0c_1^2c_2 - 27c_1^4)c_4^4 + ((144c_0^2c_2 - 6c_0c_1^2)c_3^2 + \\
&(-80c_0c_1c_2^2 + 18c_1^3c_2)c_3 + 16c_0c_2^4 - 4c_1^2c_2^3 - 10560c_0^3c_2 + 248c_0^2c_1^2)c_4^3 + (-27c_0^2c_3^4 + (18c_0c_1c_2 - \\
&4c_1^3)c_3^3 + (-4c_0c_2^3 + c_1^2c_2^2 - 9720c_0^3)c_3^2 + (10152c_0^2c_1c_2 - 682c_0c_1^3)c_3 + 4816c_0^2c_2^3 - 5428c_0c_1^2c_2^2 + \\
&1020c_1^4c_2 + 43200c_0^4)c_4^2 + (3942c_0^2c_1c_3^3 + (-4536c_0^2c_2^2 - 2412c_0c_1^2c_2 + 560c_1^4)c_3^2 + (3272c_0c_1c_2^3 - \\
&746c_1^3c_2^2 - 31320c_0^3c_1)c_3 - 576c_0c_2^5 + 144c_1^2c_2^4 - 6480c_0^3c_2^2 + 8748c_0^2c_1^2c_2 - 1700c_0c_1^4)c_4 + 162c_0^2c_2c_3^4 + \\
&(-108c_0c_1c_2^2 + 24c_1^3c_2)c_3^3 + (24c_0c_2^4 - 6c_1^2c_2^3 - 27540c_0^3c_2 + 15417c_0^2c_1^2)c_3^2 + (16632c_0^2c_1c_2^2 - 12330c_0c_1^3c_2 + \\
&2000c_1^5)c_3 - 192c_0^2c_2^4 + 248c_0c_1^2c_2^3 - 50c_1^4c_2^2 - 32400c_0^4c_2 + 540c_0^3c_1^2)c_5^2 + ((6912c_0^3c_3 - 640c_0^2c_1c_2 + \\
&144c_0c_1^3)c_4^4 + (-4464c_0^2c_1c_3^2 + (-2496c_0^2c_2^2 + 3272c_0c_1^2c_2 - 630c_1^4)c_3 - 96c_0c_1c_2^3 + 24c_1^3c_2^2 - 21888c_0^3c_1)c_4^3 + \\
&((2808c_0^2c_2 - 108c_0c_1^2)c_3^3 + (-1584c_0c_1c_2^2 + 356c_1^3c_2)c_3^2 + (320c_0c_2^4 - 80c_1^2c_2^3 - 3456c_0^3c_2 + 16632c_0^2c_1^2)c_3 + \\
&15264c_0^2c_1c_2^2 - 13040c_0c_1^3c_2 + 2250c_1^5)c_4^2 + (-486c_0^2c_3^5 + (324c_0c_1c_2 - 72c_1^3)c_3^4 + (-72c_0c_2^3 + 18c_1^2c_2^2 + \\
&21384c_0^3)c_3^3 + (-22896c_0^2c_1c_2 + 1980c_0c_1^3)c_3^2 + (-5760c_0^2c_2^3 + 10152c_0c_1^2c_2^2 - 2050c_1^4c_2 - 77760c_0^4)c_3 - \\
&640c_0c_1c_2^4 + 160c_1^3c_2^3 + 31968c_0^3c_1c_2 - 1800c_0^2c_1^3)c_4 - 6318c_0^2c_1c_3^4 + (5832c_0^2c_2^2 + 3942c_0c_1^2c_2 - 900c_1^4)c_3^3 + \\
&(-4464c_0c_1c_2^3 + 1020c_1^3c_2^2 + 15552c_0^3c_1)c_3^2 + (768c_0c_2^5 - 192c_1^2c_2^4 + 46656c_0^3c_2^2 - 31320c_0^2c_1^2c_2 + \\
&2250c_0c_1^4)c_3 - 21888c_0^2c_1c_2^3 + 15600c_0c_1^3c_2^2 - 2500c_1^5c_2 + 38880c_0^4c_1)c_5 - 1024c_0^3c_4^6 + (768c_0^2c_1c_3 + \\
&512c_0^2c_2^2 - 576c_0c_1^2c_2 + 108c_1^4)c_4^5 + ((-576c_0^2c_2 + 24c_0c_1^2)c_3^2 + (320c_0c_1c_2^2 - 72c_1^3c_2)c_3 - 64c_0c_2^4 + \\
&16c_1^2c_2^3 + 9216c_0^3c_2 - 192c_0^2c_1^2)c_4^4 + (108c_0^2c_3^4 + (-72c_0c_1c_2 + 16c_1^3)c_3^3 + (16c_0c_2^3 - 4c_1^2c_2^2 - 8640c_0^3)c_3^2 + \\
&(-5760c_0^2c_1c_2 - 120c_0c_1^3)c_3 - 4352c_0^2c_2^3 + 4816c_0c_1^2c_2^2 - 900c_1^4c_2 - 13824c_0^4)c_4^3 + (5832c_0^2c_1c_3^3 + \\
&(8208c_0^2c_2^2 - 4536c_0c_1^2c_2 + 825c_1^4)c_3^2 + (-2496c_0c_1c_2^3 + 560c_1^3c_2^2 + 46656c_0^3c_1)c_3 + 512c_0c_2^5 - 128c_1^2c_2^4 - \\
&17280c_0^3c_2^2 - 6480c_0^2c_1^2c_2 + 1500c_0c_1^4)c_4^2 + ((-4860c_0^2c_2 + 162c_0c_1^2)c_3^4 + (2808c_0c_1c_2^2 - 630c_1^3c_2)c_3^3 + \\
&(-576c_0c_2^4 + 144c_1^2c_2^3 + 3888c_0^3c_2 - 27540c_0^2c_1^2)c_3^2 + (-3456c_0^2c_1c_2^2 + 19800c_0c_1^3c_2 - 3750c_1^5)c_3 + \\
&9216c_0^2c_2^4 - 10560c_0c_1^2c_2^3 + 2000c_1^4c_2^2 + 62208c_0^4c_2 - 32400c_0^3c_1^2)c_4 + 729c_0^2c_3^6 + (-486c_0c_1c_2 + 108c_1^3)c_3^5 + \\
&(108c_0c_2^3 - 27c_1^2c_2^2 - 8748c_0^3)c_3^4 + (21384c_0^2c_1c_2 - 1350c_0c_1^3)c_3^3 + (-8640c_0^2c_2^3 - 9720c_0c_1^2c_2^2 + 2250c_1^4c_2 + \\
&34992c_0^4)c_3^2 + (6912c_0c_1c_2^4 - 1600c_1^3c_2^3 - 77760c_0^3c_1c_2 + 27000c_0^2c_1^3)c_3 - 1024c_0c_2^6 + 256c_1^2c_2^5 - \\
&13824c_0^3c_2^3 + 43200c_0^2c_1^2c_2^2 - 22500c_0c_1^4c_2 + 3125c_1^6 - 46656c_0^5.
\end{aligned}$$

References

[1] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*, UTM, Springer-Verlag, 1998.

[2] D. Cox, J. Little, and D. O'Shea, *Using Algebraic Geometry*, GTM 185, Springer-Verlag, 1998.

[3] A. Eremenko and A. Gabrielov, Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry, *Ann. of Math.*, **155** (2002), 105–129.

[4] A. Eremenko and A. Gabrielov, Elementary proof of the B. and M. Shapiro conjecture for rational functions, *preprint*.

[5] M. Fujimura and M. Taniguchi, Stratification and coordinate systems for the moduli space of rational functions, *Conform. Geom. Dyn.*, **14** (2010), 141–153.

[6] L. Goldberg Catalan numbers and branched covering by the Riemann sphere, *Adv. Math.*, **85** (1991), 129–144.

[7] M. Jeong and M. Taniguchi, Bell representation of finitely connected planar domains, *Proc. AMS.*, **131** (2003), 2325–2328.

[8] I. Scherbak, Rational functions with prescribed critical points, *GAFA*, **12** (2002), 1365–1380.

Masayo Fujimura, Department of Mathematics, National Defense Academy, Yokosuka 239-8686, JAPAN (masayo@nda.ac.jp)

Mohaby Karima, Graduate School of Humanities and Sciences, Nara Women's University, Nara 630-8506, JAPAN (iam_karima@cc.nara-wu.ac.jp)

Masahiko Taniguchi, Department of Mathematics, Nara Women's University, Nara 630-8506, JAPAN (tanig@cc.nara-wu.ac.jp)